Closed-Form Solution Of Absolute Orientation Using Unit Quaternions

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November 11, 2004

Outline

The Problem

- Given: two sets of corresponding points in different coordinate systems
- Task: find the absolute orientation of the two systems (scale, rotation, translation)
- Previous approaches:
  - Only iterative solutions
  - Only close to least-squares solution
  - Only handle three points
  - Selectively neglect constraints
- This paper:
  - Closed-form, exact least-squares solution
  - No constraints neglected

Quaternions

Quaternion Notation

$$\mathring{q} = q_0 + iq_x + jq_y + kq_z$$
$$\mathring{q} = (q_0, q_x, q_y, q_z)^T$$

Quaternions: Basic Properties

$$i^2 = -1 \qquad j^2 = -1 \qquad k^2 = -1$$
$$ij = k \qquad jk = i \qquad ki = j$$
$$ji = -k \qquad kj = -i \qquad ik = -j$$

Products of Quaternions

Quaternion Product as a Matrix–Vector Product

$$\mathring{r}\mathring{q} = \begin{pmatrix} r_0 & -r_x & -r_y & -r_z \\ r_x & r_0 & -r_z & r_y \\ r_y & r_z & r_0 & -r_x \\ r_z & -r_y & r_x & r_0 \end{pmatrix} \mathring{q} = \mathbf{R}\mathring{q}$$

$$\mathring{q}\mathring{r} = \begin{pmatrix} r_0 & -r_x & -r_y & -r_z \\ r_x & r_0 & r_z & -r_y \\ r_y & -r_z & r_0 & r_x \\ r_z & r_y & -r_x & r_0 \end{pmatrix} \mathring{q} = \bar{\mathbf{R}}\mathring{q}$$

Watch Out!

The quaternion product is not commutative!

Dot Product

Dot Product of Two Quaternions

$$\begin{aligned} \mathring{p} \cdot \mathring{q} &= p_0 q_0 + p_x q_x + p_y q_y + p_z q_z \\ \|\mathring{q}\|^2 &= \mathring{q} \cdot \mathring{q} \end{aligned}$$

Conjugate and Inverse

$$\begin{aligned} \mathring{p}^* &= q_0 - i q_x - j q_y - k q_z \\ \mathring{q}\mathring{q}^* &= q_0^2 + q_x^2 + q_y^2 + q_z^2 = \mathring{q} \cdot \mathring{q} \end{aligned}$$

A non-zero quaternion has an inverse

$$\mathring{q}^{-1} = \frac{\mathring{q}^*}{\mathring{q} \cdot \mathring{q}}$$

Useful Properties of Quaternions

Dot Products

$$(\mathring{q}\mathring{p}) \cdot (\mathring{q}\mathring{r}) = (\mathring{q} \cdot \mathring{q})(\mathring{p} \cdot \mathring{r})$$
$$(\mathring{p}\mathring{q}) \cdot \mathring{r} = \mathring{p} \cdot (\mathring{r}\mathring{q}^*)$$

Representing Vectors

Let $\mathbf{r} = (x, y, z)^T \in \mathbb{R}^3$, then

$$\mathring{r} = 0 + ix + iy + iz \ .$$

Matrices associated with such purely imaginary quaternions are skew symmetric: (go back to matrices)

$$\mathbf{R}^T = -\mathbf{R} \qquad \bar{\mathbf{R}}^T = -\bar{\mathbf{R}}$$

Representing Rotations With Quaternions

Rotation Representation Using a Unit Quaternion

$$\mathring{q} = \cos\frac{\phi}{2} + \sin\frac{\phi}{2}\underbrace{(i\omega_x + j\omega_y + k\omega_z)}_{\text{unit vector } \omega}$$

The imaginary part gives the direction of the axis of rotation. The angle is encoded into the real part and the magnitude of the imaginary part.

Rotating a Vector

Note that a vector is represented using an imaginary quaternion!

$$\mathring{r}' = \mathring{q}\mathring{r}\mathring{q}^* \qquad \mathring{r}'' = \mathring{p}\mathring{r}'\mathring{p}^* = \mathring{p}(\mathring{q}\mathring{r}\mathring{q}^*)\mathring{p}^* = \mathring{p}\mathring{q}\mathring{r}(\mathring{p}\mathring{q})^*$$

Solving the Absolute Orientation Problem

- Given: n points, each measured in a left and right coordinate system

$$\{\mathbf{r}_{l,i}\} \qquad \text{and} \qquad \{\mathbf{r}_{r,i}\}$$

- Try to find a transformation of the form

$$\mathbf{r}_r = sR(\mathbf{r}_l) + \mathbf{r}_0$$

from the left to the right coordinate system.

- There will be errors

$$\mathbf{e}_i = \mathbf{r}_{r,i} - sR(\mathbf{r}_{l,i}) - \mathbf{r}_0$$

- Minimize the sum of squares of errors

$$\sum_{i=1}^{n} \|\mathbf{e}_i\|^2$$

Translation

Working Relative to the Centroids

$$\bar{\mathbf{r}}_l = \frac{1}{n}\sum_{i=1}^{n} \mathbf{r}_{l,i} \qquad \bar{\mathbf{r}}_r = \frac{1}{n}\sum_{i=1}^{n} \mathbf{r}_{r,i}$$

$$\mathbf{r}'_{l,i} = \mathbf{r}_{l,i} - \bar{\mathbf{r}}_l \qquad \mathbf{r}'_{r,i} = \mathbf{r}_{r,i} - \bar{\mathbf{r}}_r$$

- Rewrite the error term:

$$\mathbf{e}_i = \mathbf{r}'_{r,i} - sR(\mathbf{r}'_{l,i}) - \mathbf{r}'_0 \qquad \text{where } \mathbf{r}'_0 = \mathbf{r}_0 - \bar{\mathbf{r}}_r + sR(\bar{\mathbf{r}}_l)$$

- The sum of squares of errors becomes

$$\sum_{i=1}^{n} \|\mathbf{e}_i\|^2 = \sum_{i=1}^{n} \|\mathbf{r}'_{r,i} - sR(\mathbf{r}'_{l,i}) - \mathbf{r}'_0\|^2$$

Translation

- Decompose the sum:

$$\sum_{i=1}^{n} \|\mathbf{r}'_{r,i} - sR(\mathbf{r}'_{l,i}) - \mathbf{r}'_0\|^2$$
$$= \underbrace{\sum_{i=1}^{n} \|\mathbf{r}'_{r,i} - sR(\mathbf{r}'_{l,i})\|^2}_{\text{independent of translation}} - \underbrace{2\mathbf{r}'_0 \cdot \left(\sum_{i=1}^{n} \mathbf{r}'_{r,i} - sR(\mathbf{r}'_{l,i})\right)}_{=0} + n\|\mathbf{r}'_0\|^2$$

- The total error is minimized with $\mathbf{r}'_0 = 0$, i.e. $\mathbf{r}_0 = \bar{\mathbf{r}}_r - sR(\bar{\mathbf{r}}_l)$.
- Remaining error term: $\mathbf{e}_i = \mathbf{r}'_{r,i} - sR(\mathbf{r'}_{l,i})$. Now minimize

$$\sum_{i=1}^{n} \|\mathbf{r}'_{r,i} - sR(\mathbf{r}'_{l,i})\|^2 \ .$$

Finding the Scale

- Decompose the sum:

$$\begin{aligned} & \sum_{i=1}^{n} \|\mathbf{r}'_{r,i} - sR(\mathbf{r}'_{l,i})\|^2 \\ = \quad & \sum_{i=1}^{n} \|\mathbf{r}'_{r,i}\|^2 - 2s \sum_{i=1}^{n} \mathbf{r}'_{r,i} \cdot R(\mathbf{r}'_{l,i}) + s^2 \sum_{i=1}^{n} \|\mathbf{r}'_{l,i}\|^2 \\ = \quad & S_r - 2sD + s^2 S_l \end{aligned}$$

- Complete the square:

$$\underbrace{(s\sqrt{S_l} - D/\sqrt{S_l})^2}_{=0} + \underbrace{(S_r S_l - D^2)/S_l}_{\text{independent of scale}}$$

- Best scale is $s = D/S_l$.

Finding the Scale

$$s = D/S_l = \frac{\sum_{i=1}^{n} \mathbf{r}'_{r,i} \cdot R(\mathbf{r}'_{l,i})}{\sum_{i=1}^{n} \|\mathbf{r}'_{l,i}\|^2}$$

- This scale factor is not symmetric! When going from the right to the left system, we get

$$\bar{s} \neq \frac{1}{s}$$

- Use a symmetrical expression for the error term instead:

$$\mathbf{e}'_i = \frac{1}{\sqrt{s}}\mathbf{r}'_{r,i} - \sqrt{s}R(\mathbf{r}'_{l,i}) = \frac{\mathbf{e}_i}{\sqrt{s}}$$

Finding the Scale

- Total error is then

$$\frac{1}{s}\sum_{i=1}^{n}\|\mathbf{r}'_{r,i}\|^2 - 2\sum_{i=1}^{n}\mathbf{r}'_{r,i}\cdot R(\mathbf{r}'_{l,i}) + s\sum_{i=1}^{n}\|\mathbf{r}'_{l,i}\|^2 = \frac{1}{s}S_r - 2D - sS_l$$

- Complete the square (slightly different from Horn p. 632)

$$\underbrace{(\sqrt{s}\sqrt{S_l} - \sqrt{S_r}/\sqrt{s})^2}_{=0} + \underbrace{2(\sqrt{S_l S_r} - D)}_{\text{independent of scale}}$$

- Best scale is now independent of rotation:

$$s = \sqrt{S_r/S_l} = \sqrt{\sum_{i=1}^{n}\|\mathbf{r}'_{r,i}\|^2 \Big/ \sum_{i=1}^{n}\|\mathbf{r}'_{l,i}\|^2}$$

Rotation

- Remaining error term to be minimized:

$$(S_r S_l - D^2)/S_l \qquad \text{or} \qquad 2(\sqrt{S_l S_r} - D)$$

(asymmetric) (symmetric)

I.e., maximize

$$D = \sum_{i=1}^{n} \mathbf{r}'_{r,i} \cdot R(\mathbf{r}'_{l,i})$$

- Use the quaternion representation: find the unit quaternion $\mathring{q}$ that maximizes

$$D = \sum_{i=1}^{n} (\mathring{q}\mathring{r}'_{l,i}\mathring{q}^*) \cdot \mathring{r}'_{r,i}$$

Rotation

Using $(\mathring{p}\mathring{q}) \cdot \mathring{r} = \mathring{p} \cdot (\mathring{r}\mathring{q}^*)$ from earlier, rewrite the term as

$$\begin{aligned}
\sum_{i=1}^{n} (\mathring{q}\mathring{r}'_{l,i}\mathring{q}^*) \cdot \mathring{r}'_{r,i} &= \sum_{i=1}^{n} (\mathring{q}\mathring{r}'_{l,i}) \cdot (\mathring{r}'_{r,i}\mathring{q}) \\
&= \sum_{i=1}^{n} (\bar{\mathbf{R}}_{l,i}\mathring{q}) \cdot (\mathbf{R}_{r,i}\mathring{q}) \\
&= \sum_{i=1}^{n} \mathring{q}^T \bar{\mathbf{R}}_{l,i}^T \mathbf{R}_{r,i} \mathring{q} \\
&= \mathring{q}^T \left(\sum_{i=1}^{n} \bar{\mathbf{R}}_{l,i}^T \mathbf{R}_{r,i} \right) \mathring{q} \\
&= \mathring{q}^T \left(\sum_{i=1}^{n} N_i \right) \mathring{q} = \mathring{q}^T N \mathring{q}
\end{aligned}$$

Rotation

Utilize the 3×3 matrix

$$M := \sum_{i=1}^{n} \mathbf{r}'_{l,i} \mathbf{r}'^{T}_{r,i} := \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}$$

for a convenient representation of

$$N = \begin{pmatrix} S_{xx} + S_{yy} + S_{zz} & S_{yz} - S_{zy} & S_{zx} - S_{xz} & S_{xy} - S_{yx} \\ S_{yz} - S_{zy} & S_{xx} - S_{yy} - S_{zz} & S_{xy} + S_{yx} & S_{zx} + S_{xz} \\ S_{zx} - S_{xz} & S_{xy} + S_{yx} & -S_{xx} + S_{yy} - S_{zz} & S_{yz} + S_{zy} \\ S_{xy} - S_{yx} & S_{zx} + S_{xz} & S_{yz} + S_{zy} & -S_{xx} - S_{yy} + S_{zz} \end{pmatrix}$$

Note that N is symmetric, and $\operatorname{trace}(N) = 0$. This contains all information to solve the least-squares problem for rotation!

Rotation

- The unit quaternion that maximizes $\mathring{q}^T N \mathring{q}$ is the eigenvector to the most positive eigenvalue of N
- To find this eigenvalue solve the quartic obtained from

$$\det(N - \lambda I) = 0$$

  Use e.g. Ferrari's method.
- For the eigenvalue λ_m, the eigenvector $\mathring{e}_m$ is found by solving

$$(N - \lambda_m I)\mathring{e}_m = 0$$

- A lot easier nowadays using SVD...

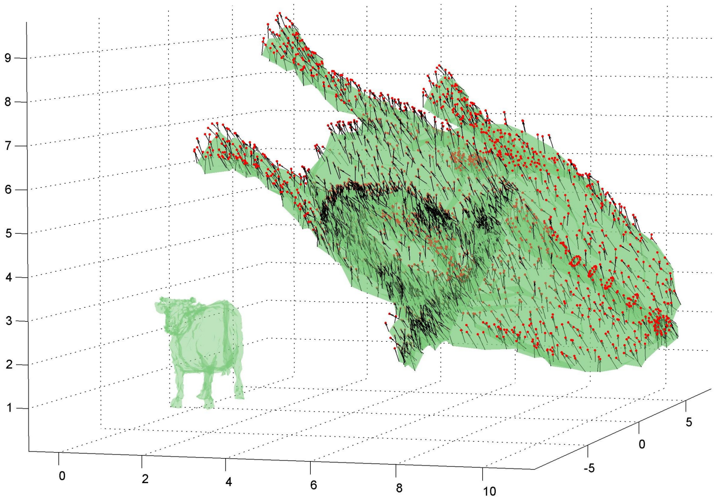
The Algorithm

1. Find the centroids $\bar{\mathbf{r}}_l, \bar{\mathbf{r}}_r$, substract from measurements
2. For each pair $\left(\mathbf{r}'_l = (x'_l, y'_l, z'_l), \mathbf{r}'_r = (x'_r, y'_r, z'_r)\right)$ compute all possible products $x'_l x'_r, x'_l y'_r, \ldots, z'_l z'_r$ and add up to obtain $S_{xx}, S_{xy}, \ldots, S_{zz}$
3. Compute elements of N
4. Calculate the coefficients of the quartic and solve quartic
5. Pick the most positive root and obtain corresponding eigenvector. The quaternion representing the rotation is a unit vector in the same direction
6. Compute the scale
7. Compute the translation

Special Cases and Extensions

- If points are coplanar (e.g. only three points given), the calculation simplifies greatly
- Can also use weighted errors:
  - Minimize weighted sum of errors: $\sum_{i=1}^{n} w_i \|\mathbf{e}_i\|^2$
  - Calculate weighted centroids: $\mathbf{r}_l = \sum_{i=1}^{n} w_i \mathbf{r}_{l,i} \Big/ \sum_{i=1}^{n} w_i$ etc.
  - Change scale factor calculation: $S_r = \sum_{i=1}^{n} w_i \|\mathbf{r}'_{r,i}\|^2$ etc.
  - Change components of matrix for rotation recovery: $S_{xx} = \sum_{i=1}^{n} w_i x'_{l,i} x'_{r,i}$

MatLab Implementation



Summary

- I presented a closed-form solution to the absolute orientation problem
- Given a mechanism for SVD or eigenvalue computation, the solution is straightforward
- Non-Gaussian noise, statistical outliers are not handled well
- Numerical stability?

- Project
  - Combine with registration?
  - Use for non-rigid motion? E.g., by doing absolute orientation on local point sets?

References

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