Topic 8 | Calculating Potentials Due to Charged Conductors

A Case Study in Computer Analysis

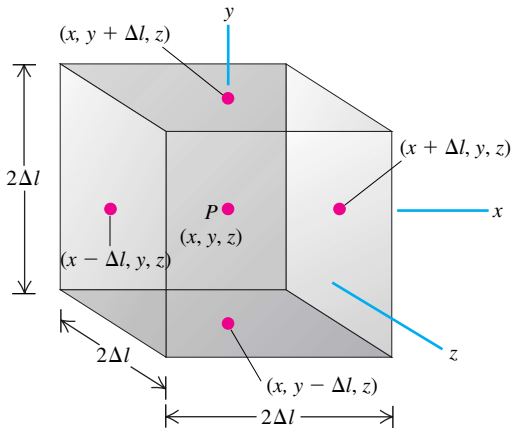
The techniques we have used in this chapter for calculating potentials and electric fields are useful only if the charge distribution is known. In many practical situations, however, the charge distribution is *not* known; instead, the value of the potential is known on the boundaries of a region. For instance, in an electrostatic situation the surface of a conductor is always an equipotential surface, but the distribution of charge on the surface is in general not uniform and is not readily calculated by the techniques we've encountered so far. Consider a region of space enclosed by one or more conductors maintained at fixed potentials (for example, by a battery). How can we determine the potential as a function of position in this region?

The key to solving this problem is to use the following fact about the electric potential: *In a region where there is no charge, the value of the potential at a given point is equal to the average of the potential values at surrounding points.* We'll prove this statement by using Gauss's law in conjunction with:

$$E_x = \frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = \frac{\partial V}{\partial z} \quad \begin{pmatrix}\text{components of } \vec{\boldsymbol{E}} \\ \text{in terms of } V\end{pmatrix} \tag{T8.1}$$

This equation gives the electric field components in terms of partial derivatives of the potential.

We'll confine our discussion to situations in which the potential depends only on two coordinates, x and y. An example is the potential due to a long charged cylinder. The potential at a point depends only on the point's coordinates in a plane perpendicular to the axis of the cylinder, not on the coordinate z along the axis. For such a two-dimensional situation, consider a point P at coordinates (x, y, z), and enclose it by a Gaussian surface in the shape of a cubical box of side $2\Delta l$ centered on P (Fig. T8.1). If there is no charge in the volume enclosed by the box, the total electric flux Φ_E through the box is equal to zero. From Eq. (T8.1) the z-component of the electric field $E_z = -\partial V/\partial z$ equals zero because the potential V is not a function of z. Hence there is no flux through the two faces of the Gaussian surface that are parallel to the xy-plane. Since the box is small, to a good



T8.1 In a region where there is no charge, the value of the potential at a point P equals the average of the potential values at points surrounding P.

approximation the flux through each of the other four faces of the box is equal to the product of the normal component of $\vec{E}$ at the center of each face and the area $(2\Delta l)^2$ of each face. The total flux (equal to zero) through the box can then be expressed as

$$\Phi_E = E_x(x + \Delta l, y, z)(2\Delta l)^2 + [-E_x(x - \Delta l, y, z)](2\Delta l)^2 \qquad \text{(T8.2)}$$

$$+ E_y(x, y + \Delta l, z)(2\Delta l)^2 + [-E_y(x, y - \Delta l, z)](2\Delta l)^2 = 0$$

Using Eq. (T8.1), we can write the electric-field components to the same approximation as

$$E_x(x + \Delta l, y, z) = -\frac{\partial V(x + \Delta l, y)}{\partial x} = -\frac{V(x + \Delta l, y) - V(x, y)}{\Delta l}$$

$$E_x(x + \Delta l, y, z) = -\frac{\partial V(x - \Delta l, y)}{\partial x} = -\frac{V(x,y) - V(x - \Delta l, y)}{\Delta l}$$

$$E_x(x, y + \Delta l, z) = -\frac{\partial V(x, y + \Delta l)}{\partial x} = -\frac{V(x, y + \Delta l) - V(x, y)}{\Delta l}$$

$$E_x(x, y - \Delta l, z) = -\frac{\partial V(x, y - \Delta l)}{\partial y} = -\frac{V(x, y) - V(x, y - \Delta l)}{\Delta l} \qquad \text{(T8.3)}$$

Substituting Eqs. (T8.3) into Eq. (T8.2) and dividing through by 4 Δl, we obtain

$$-[V(x + \Delta l, y) - V(x, y)] + [V(x, y) - V(x - \Delta l, y)]$$

$$-[V(x, y + \Delta l) - V(x, y)] + [V(x, y) - V(x, y - \Delta l) = 0$$

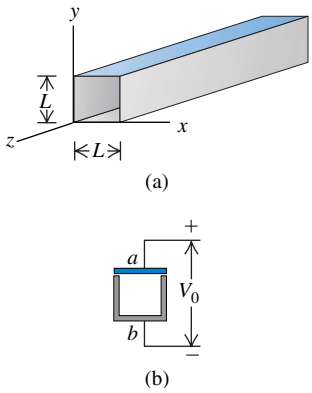
If we solve this for $V(x, y)$, the potential at point P, we find

$$V(x, y) = \frac{1}{4}[V(x + \Delta l, y) + V(x - \Delta l, y) + V(x, y + \Delta l) + V(x, y - \Delta l)] \qquad \text{(T8.4)}$$

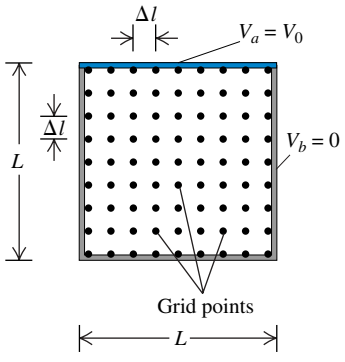
In words, the value of the potential at P is the average of the potential values at the points surrounding P. This statement becomes exact in the limit that Δl becomes infinitesimally small.

To see how to use Eq. (T8.4) to calculate the potential due to a set of charged conductors, let's consider a specific situation. Figure T8.2a shows a hollow conducting box with a square cross section and with a long axis parallel to the z-axis. The length of the box is very much greater than the dimension L. The top of the box, labeled a, is insulated from the other three sides, collectively labeled b; this is done by having the top be a separate piece of metal with a small gap between it and the vertical sides of the box (Fig. T8.2b). A fixed potential difference V_0 is maintained between segments a and b of the box. We choose the potentials of the lower segments to be $V_b = 0$, so the potential of the upper segment is $V_a = V_0$. As a result of the potential difference, there is a positive charge on a (the higher-potential conductor) and a negative charge on b (the lower-potential conductor).

Our goal is to find the potential V at all points in the interior volume of the box. Because the box is long, the potential inside the box is to a good approximation a function of x and y only. We imagine making a rectangular grid of points inside the box, separated by a distance Δl (Fig. T8.3). The outermost points of the grid are on



T8.2 (a) A conducting box whose z-dimension is much longer than the dimension L along the x- and y-axes. (b) A view down the z-axis of the box, showing the two segments into which the box is divided and the potential difference between the segments.



T8.3 A view down the long axis of the conducting box showing a rectangular grid of points separated by Δl.

the conductor surfaces themselves. Equation (T8.4) then relates the potential at different grid points to each other; since the potentials of the conductors are specified, we can determine the potential at each grid point in the empty interior of the box.

Complications arise because Eq. (T8.4) relates the values of the potential at four different grid points. The value of V is unknown at each grid point in the interior of the box, and each such point is surrounded by two or three other interior points at which V is also unknown. So Eq. (T8.4) can't be used to solve for the values of the potential at these interior points in a single step. Instead, we have to use an *iterative* method; we will make a series of successive approximations to find a set of values of V at the interior grid points such that Eq. (T8.4) is satisfied at every point. The procedure that we follow to do this is called the *relaxation method.*

Here's a skeleton of a computer program to carry out this calculation.

Step 1: Choose a *positive* value of the potential difference V_0. (The procedure described below runs into trouble if $V_0 < 0$.)

Step 2: Choose the number m of grid points across or down the region shown in Fig. T8.3. The total number of grid points is then m^2, and the total number of grid points in the interior of the box is $(m - 2)^2$. Values of m between 10 and 40 work well. Large values of m require lengthy calculation; small values of m give low resolution.

Step 3: Let (j, k) be a pair of integer indexes that identify a particular grid point and its location in the grid (the jth column and kth row in the grid); each ranges from 1 to m.

Step 4: Begin a loop on k, from $k = 1$ to $k = m$ (successive rows of grid points).

Step 5: Begin a loop on j, from $j = 1$ to $j = m$ (successive columns of grid points).

Step 6: Assign an initial value to the potential $V(j, k)$ for each grid point. If $k = 1$ (the top row, corresponding to the surface of the upper conductor a in Fig. T8.3), set $V(j, k)$ equal to V_0. If $j = 1$, $j = m$, or $k = m$, set $V(j, k)$ equal to zero; these correspond to the left, right, and bottom surfaces, respectively, of the lower conductor b in Fig. T8.3. For all other values of (j, k), corresponding to grid points in the empty interior of the box, assign an arbitrary value of $V(j, k)$. The closer this arbitrary value is to the actual value, the fewer iterations will be required to obtain a good solution. But any value greater than 0 and less than V_0 will do. A good choice for the arbitrary value might be $0.5V_0$ for all interior grid points. Do *not* use $V = 0$; this choice will cause problems in Step 13.

Step 7: End of loops over j and k.

Step 8: Specify the desired accuracy of the results, expressed as a fractional uncertainty. The program is about to begin iterating to find a solution for $V(j, k)$ at grid points inside the box, and the iteration will stop when the fractional change in the values from one iteration to the next is less than the desired accuracy. Reasonable values are from 0.01 to 0.001; the smaller the value chosen, the more iterations will be required.

Step 9: Define a quantity δ (lowercase "delta") and set it equal to zero.

Step 10: Again begin a loop on k, this time from $k = 2$ to $k = m - 1$ (that is, over interior points only).

Step 11: Begin a loop on j, this time from $j = 2$ to $j = m - 1$ (interior points only).

Step 12: Compute new values of $V_{\text{new}}(j, k)$ for each interior grid point, using Eq. (T8.4). (The loops over j and k do not include the values 1 and m

because the values of the potential on the conductor surfaces are fixed.) That is, let

$$V_{\text{new}}(j, k) = \frac{1}{4}[V(j + 1, k) + V(j - 1, k) + V(j, k + 1) + V(j, k - 1)]$$

Referring to Fig. T8.3, this means that the new value of the potential at (j, k) is the average of the old values of V at the four grid points immediately to the right, the left, below, and above.

Step 13: Calculate the fractional change in potential $\Delta V/V$ from the old value to the new value, and take its absolute value:

$$\frac{\Delta V}{V} = \left| \frac{V_{\text{new}}(j, k) - V(j, k)}{V(j, k)} \right|$$

A division-by-zero error will result if $V(j, k) = 0$; this is the reason for our cautionary note at the end of Step 6. If $\Delta V/V$ is greater than δ, set δ equal to $\Delta V/V$.

Step 14: Replace the old value of V by the new value. That is, set $V(j, k)$ equal to $V_{\text{new}}(j, k)$.

Step 15: End of loops over j and k.

Step 16: If the value of δ is greater than the desired accuracy specified in Step 8, there is at least one interior grid point at which the change in V during the iteration just completed was greater than the desired accuracy. Hence one or more additional iterations are required, and the program must return to Step 10. If the value of δ is less than or equal to the desired accuracy, the current values of $V(j, k)$ are adequate, and the program may proceed to Step 17.

Step 17: Print out the potential for each grid point.

Step 18: END.

Figure T8.4 shows the result of this calculation.

The program described above is actually easier to implement with a spreadsheet than with a programming language such as BASIC or Pascal. Most popular spreadsheets permit iterative calculations and will automatically go through the steps described above. To carry out the calculation, choose a square grid of spreadsheet cells, enter the values of the conductor potentials in the cells on the

1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.00	0.49	0.68	0.76	0.80	0.82	0.82	0.80	0.76	0.68	0.49	0.00
0.00	0.28	0.46	0.57	0.62	0.65	0.65	0.62	0.57	0.46	0.28	0.00
0.00	0.18	0.33	0.42	0.48	0.50	0.50	0.48	0.42	0.32	0.18	0.00
0.00	0.13	0.23	0.31	0.36	0.38	0.38	0.36	0.31	0.23	0.13	0.00
0.00	0.09	0.17	0.23	0.27	0.29	0.29	0.27	0.23	0.17	0.09	0.00
0.00	0.06	0.12	0.17	0.20	0.21	0.21	0.20	0.17	0.12	0.06	0.00
0.00	0.05	0.09	0.12	0.14	0.15	0.15	0.14	0.12	0.09	0.05	0.00
0.00	0.03	0.06	0.08	0.10	0.11	0.11	0.10	0.08	0.06	0.03	0.00
0.00	0.02	0.04	0.05	0.06	0.07	0.07	0.06	0.05	0.04	0.02	0.00
0.00	0.01	0.02	0.02	0.03	0.03	0.03	0.03	0.02	0.02	0.01	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

T8.4 Potentials after 75 iterations for the situation shown in Fig. T8.3, with $V_0 = 1.00$ V. The grid points on the conductor surfaces are shown shaded for clarity. For this grid, $m = 12$. The potential decreases as you move away from the higher-potential top surface and toward the lower-potential surfaces of the sides and bottom.

periphery of the grid, and enter the formula corresponding to Eq. (T8.4) into each interior cell. The results shown in Fig. T8.4 were obtained with a spreadsheet.

It's easy to modify the above program to treat other types of conducting boundaries. The values of the potential on the conducting surfaces can be chosen at will. The box can be changed from a square to a rectangle by having a different number of rows than columns. The box can also have shapes other than rectangular. Various suggestions are included in the exercises.

Once the values of the potential have been calculated for all grid points, the electric field components E_x and E_y can be calculated as well using equations similar to Eq. (T8.3). At a point (x, y) in the interior of the box, we can find E_x and E_y using the approximate results

$$E_x(x, y) = -\frac{\partial V(x, y)}{\partial x} \approx -\frac{V(x + \Delta l, y) - V(x - \Delta l, y)}{2\Delta l}$$

$$E_y(x, y) = -\frac{\partial V(x, y)}{\partial y} \approx -\frac{V(x + y, \Delta l) - V(x, y - \Delta l)}{2\Delta l} \qquad \text{(T8.5)}$$

The magnitude of the field is

$$E = \sqrt{E_x^{\,2} + E_y^{\,2}}. \qquad \text{(T8.6)}$$

To indicate the direction of $\vec{E}$ at each interior grid point, the program can be instructed to draw a line of length c between the two points with coordinates

$$(x, y) \quad \text{and} \quad \left(x + \frac{cE_x}{E}, y + \frac{cE_y}{E}\right) \qquad \text{(T8.7)}$$

You can verify that the distance between these points is c and that the direction of the line is the same as that of $\vec{E}$ at (x, y).

To draw lines indicating the direction of $\vec{E}$ at each interior grid point, insert the following additional steps between Steps 17 and 18 in the above program skeleton. (*Note:* This may be easier to implement by using BASIC or Pascal than with a spreadsheet.)

Step 17A: Choose the screen coordinates (addresses of pixels on the screen) within which the grid points and direction lines of $\vec{E}$ will be drawn, that is, values of $x_{\min}$, $y_{\min}$, $x_{\max}$, and $y_{\max}$. These values are determined by the characteristics of your computer. For example, a Macintosh with a 14-inch monitor has $x_{\min} = 0$, $x_{\max} = 639$, $y_{\min} = 0$, and $y_{\max} = 479$. Some languages let you change the range of the screen coordinates. In BASIC, for example, the WINDOW command lets you choose any range of x and y you like; it does the conversion to screen coordinates for you.

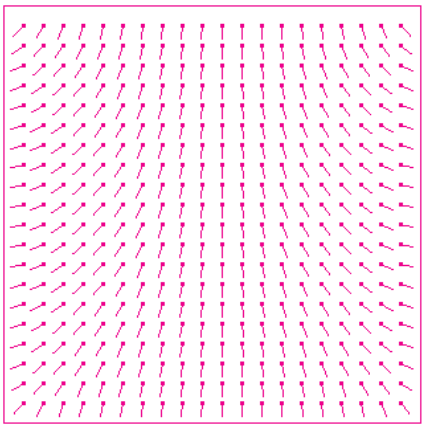
Step 17B: Choose the distance Δl (the spacing, in pixels, between grid points on the screen) so that all the grid points will fit on the screen. For instance, if the number of grid rows and columns is $m = 20$, choose Δl to be 1/20 of the *smaller* of $(x_{\max} - x_{\min})$ and $(y_{\max} - y_{\min})$. Set the line length c equal to Δl.

Step 17C: Begin a loop on k from $k = 2$ to $k = m - 1$ (interior points only).

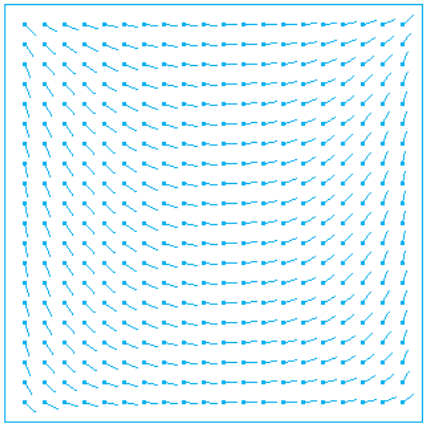
Step 17D: Begin a loop on j from $j = 2$ to $j = m - 1$ (interior points only).

Step 17E: Calculate the x- and y-coordinates of the screen position corresponding to the current grid point:

$$x = x_{\min} + j\Delta l \quad \text{and} \quad y = y_{\min} + k\Delta l$$



T8.5 Electric-field map for the situation shown in Fig. T8.3. Each grid point is marked with a dot. For this grid, $m = 22$. The field is directed away from the positive charges on the top surface at $V = V_0$ and toward the negative charges on the sides and bottom at $V = 0$.



T8.6 Equipotential line segments for the situation shown in Fig. T8.3. Each grid point is marked with a dot. For this grid, $m = 22$. As you approach the conducting surfaces at the top, left, bottom, and right, the equipotentials become nearly parallel to the surfaces.

Step 17F: Calculate the electric-field components at the interior grid point (j, k) using Eq. (8.5):

Step 17G: If $E_x = 0$ and $E_y = 0$, do not draw a line for the direction of $\vec{E}$ (since the field is zero); instead, go on to the next grid point. Otherwise, find E using Eq. (T8.6).

$$E_x = -\frac{V(j+1,k) - V(j-1,k)}{2\Delta l} \quad \text{and} \quad E_y = -\frac{V(j,k+1) - V(j,k-1)}{2\Delta l}$$

Step 17H: Draw a line between the points given by Eq. (T8.7).

Step 17I: End of loops over j and k.

Figure T8.5 shows the result of this calculation for the situation depicted in Fig. T8.3.

You may want to refine the calculation to draw at each grid point a vector with length proportional to the field magnitude E; this shows both the magnitude and the direction of the field. To do this, simply replace the parameter c in Step 17H by a multiple of E. You may have to experiment to find the appropriate scale factor for the best picture.

By a simple modification of the technique used to create the field map, we can map the equipotential lines in the xy-plane (the cross section in this plane of the equipotential *surfaces*). We use the idea that when an equipotential crosses a field line, the two are perpendicular. To map the equipotentials, we draw a line at each grid point in a direction perpendicular to the field line at that point. From analytic geometry, if two lines in a plane are perpendicular, their slopes are negative reciprocals of each other. The slope of the field line at each point is E_y/E_x, so the slope of the equipotential at the same point is $-E_x/E_y$. Referring to Eq. (T8.7), which we used to draw lines with length c parallel to the field, we see that we need only replace E_x by E_y and E_y by $-E_x$ in this expression to get a line perpendicular to $\vec{E}$ at the point. That is, in Step 17H we use, instead of Eq. (T8.7), the following:

$$(x, y) \quad \text{and} \quad \left(x + \frac{cE_y}{E}, y - \frac{cE_x}{E}\right) \tag{T8.8}$$

Figure T8.6 shows the result of such a calculation. Compare this to the field map shown in Fig. T8.5. Note that this procedure doesn't plot actual equipotential curves; rather, it plots segments of various curves. But these give us a good idea of the shapes of the equipotentials.

Finally, we note that the relaxation-method approach is just another case in which it is easiest to find the potential first, then determine the electric field from the potential.