Chapter 12

Foundations of Fluid Dynamics

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Please send comments, suggestions, and errata via email to kip@tapir.caltech.edu or on paper to Kip Thorne, 130-33 Caltech, Pasadena CA 91125

12.1   Overview

Having studied elasticity theory, we now turn to a second branch of continuum mechanics: *fluid dynamics*. Three of the four states of matter (gases, liquids and plasmas) can be regarded as fluids and so it is not surprising that interesting fluid phenomena surround us in our everyday lives. Fluid dynamics is an experimental discipline and much of what has been learned has come in response to laboratory investigations. Fluid dynamics finds experimental application in engineering, physics, biophysics, chemistry and many other fields. The observational sciences of oceanography, meteorology, astrophysics and geophysics, in which experiments are less frequently performed, are also heavily reliant upon fluid dynamics. Many of these fields have enhanced our appreciation of fluid dynamics by presenting flows under conditions that are inaccessible to laboratory study.

Despite this rich diversity, the fundamental principles are common to all of these applications. The fundamental assumption which underlies the governing equations that describe the motion of fluid is that the length and time scales associated with the flow are long compared with the corresponding microscopic scales, so the continuum approximation can be invoked. In this chapter, we will derive and discuss these fundamental equations. They are, in some respects, simpler than the corresponding laws of elastodynamics. However, as with particle dynamics, simplicity in the equations does not imply that the solutions are simple, and indeed they are not! One reason is that there is no restriction that fluid displacements be small (by constrast with elastodynamics where the elastic limit keeps them small), so most fluid phenomena are immediately nonlinear.

Relatively few problems in fluid dynamics admit complete, closed-form, analytic solutions, so progress in describing fluid flows has usually come from the introduction of clever physical "models" and the use of judicious mathematical approximations. In more recent years numerical fluid dynamics has come of age and in many areas of fluid mechanics, finite

difference simulations have begun to complement laboratory experiments and measurements.

Fluid dynamics is a subject where considerable insight accrues from being able to visualize the flow. This is true of fluid experiments where much technical skill is devoted to marking the fluid so it can be photographed, and numerical simulations where frequently more time is devoted to computer graphics than to solving the underlying partial differential equations. We shall pay some attention to flow visualization. The reader should be warned that obtaining an analytic solution to the equations of fluid dynamics is not the same as understanding the flow; it is usually a good idea to sketch the flow pattern at the very least, as a tool for understanding.

We shall begin this chapter in Sec. 12.2 with a discussion of the physical nature of a fluid: the possibility to describe it by a piecewise continuous density, velocity, and pressure, and the relationship between density changes and pressure changes. Then in Sec. 12.3 we shall discuss hydrostatics (density and pressure distributions of a static fluid in a static gravitational field); this will parallel our discussion of elastostatics in Chap. 10. Following a discussion of atmospheres, stars and planets, we shall explain the microphysical basis of Archimedes principle.

Our foundation for moving from hydrostatics to hydrodynamics will be conservation laws for mass, momentum and energy. To facilitate that transition, in Sec. 12.4 we shall examine in some depth the physical and mathematical origins of these conservation laws in Newtonian physics.

The stress tensor associated with most fluids can be decomposed into an isotropic pressure and a viscous term linear in the rate of shear or velocity gradient. Under many conditions the viscous stress can be neglected over most of the flow and the fluid is then called *ideal* or *inviscid*. We shall study the laws governing ideal flows in Sec. 12.5. After deriving the relevant conservation laws and equation of motion, we shall derive and discuss the Bernoulli principle (which relies on ideality) and show how it can simplify the description of many flows. In flows for which the speed neither approaches the speed of sound, nor the gravitational escape velocity, the fractional changes in fluid density are relatively small. It can then be a good approximation to treat the fluid as *incompressible* and this leads to considerable simplification, which we also study in Sec. 12.5. As we shall see, incompressibility can be a good approximation not just for liquids which tend to have large bulk moduli, but also, more surprisingly, for gases.

In Sec. 12.6 we augment our basic equations with terms describing the action of the viscous stresses. This allows us to derive the famous Navier-Stokes equation and to illustrate its use by analyzing pipe flow. Much of our study of fluids in future chapters will focus on this Navier Stokes equation.

In our study of fluids we shall often deal with the influence of a uniform gravitational field, such as that on earth, on lengthscales small compared to the earth's radius. Occasionally, however, we shall consider inhomogeneous gravitational fields produced by the fluid whose motion we study. For such situations it is useful to introduce gravitational contributions to the stress tensor and energy density and flux. We present and discuss these in a box, Box 12.2, where they will not impede the flow of the main stream of ideas.

12.2 The Macroscopic Nature of a Fluid: Density, Pressure, Flow velocity

The macroscopic nature of a fluid follows from two simple observations.

The first is that in most flows the macroscopic continuum approximation is valid: Because, in a fluid, the molecular mean free paths are small compared to macroscopic length-scales, we can define a mean local velocity $\mathbf{v}(\mathbf{x}, t)$ of the fluid's molecules, which varies smoothly both spatially and temporally; we call this the fluid's velocity. For the same reason, other quantities that characterize the fluid, e.g. the density $\rho(\mathbf{x}, t)$, also vary smoothly on macroscopic scales. Now, this need not be the case everywhere in the flow. The exception is a shock front, which we shall study in Chap. 16; there the flow varies rapidly, over a length of order the collision mean free path of the molecules. In this case, the continuum approximation is only piecewise valid and we must perform a matching at the shock front. One might think that a second exception is a turbulent flow where, it might be thought, the average molecular velocity will vary rapidly on whatever length scale we choose to study all the way down to intermolecular distances, so averaging becomes problematic. As we shall see in Chap. 14, this is not the case; in turbulent flows there is generally a length scale far larger than intermolecular distances within which the flow varies smoothly.

The second observation is that fluids do not oppose a steady shear strain. This is easy to understand on microscopic grounds as there is no lattice to deform and the molecular velocity distribution remains isotropic in the presence of a static shear. By kinetic theory considerations (Chap. 2), we therefore expect that the stress tensor $\mathbf{T}$ will be isotropic in the local rest frame of the fluid (i.e., in a frame where $\mathbf{v} = 0$). This allows us to write $\mathbf{T} = P\mathbf{g}$ in the local rest frame, where P the fluid's pressure and $\mathbf{g}$ is the metric (with Kronecker delta components, $g_{ij} = \delta_{ij}$).

Now suppose that we have a fluid element with pressure P and density ρ and it undergoes a small isotropic expansion with $\Theta = -\delta\rho/\rho$ [cf. Eq. (11.3)]. This expansion will produce a pressure change

$$\delta P = -K\Theta ,\tag{12.1}$$

where K is the bulk modulus, or equivalently a change in the stress tensor $\delta\mathbf{T} = -K\Theta\mathbf{g} = \delta P\mathbf{g}$. It is convenient in fluid mechanics to use a different notation: We introduce a dimensionless parameter $\Gamma \equiv K/P$ where P is the unperturbed pressure, and write $\delta P = -K\Theta = K\delta\rho/\rho = \Gamma P\delta\rho/\rho$; i.e.,

$$\frac{\delta P}{P} = \Gamma\frac{\delta\rho}{\rho}\tag{12.2}$$

.

The value of Γ depends on the physical situation. If the fluid is an ideal gas [so $P = \rho k_B T/\mu m_p$ in the notation of Box 12.1, Eq. (4)] and the temperature is being held fixed by thermal contact with some heat source as the density changes, then $\delta P \propto \delta\rho$ and $\Gamma = 1$. Alternatively, and much more commonly, the fluid's entropy might remain constant because no significant heat can flow in or out of a fluid element in the time for the density change to take place. In this case it can be shown using the laws of thermodynamics (Chap. 4) that $\Gamma = \gamma = C_P/C_V$, where C_P, C_V are the specific heats at constant pressure and volume. For the moment, though, we shall just assume that we have a prescription for relating changes
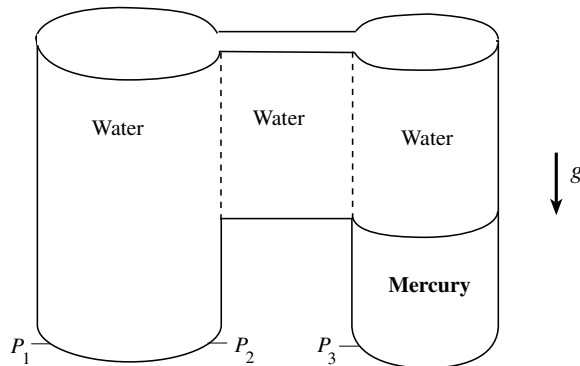
Fig. 12.1: Elementary demonstration of the principle of hydrostatic equilibrium. Water and mercury, two immisicible fluids of different density, are introduced into a container with two chambers as shown. The pressure at each point on the bottom of the container is equal to the weight per unit area of the overlying fluids. The pressures P_1 and P_2 at the bottom of the left chamber are equal, but because of the density difference between mercury and water, they differ from the pressure P_3 at the bottom of the right chamber.

in the density to corresponding changes in the pressure, and correspondingly we know the value of Γ. (See Box 12.1 for further discussion of thermodynamic aspects of fluid dynamics.)

12.3 Hydrostatics

Just as we began our discussion of elasticity with a treatment of elastostatics, so we will introduce fluid mechanics by discussing hydrostatic equilibrium.

The *equation of hydrostatic equilibrium* for a fluid at rest in a gravitational field $\boldsymbol{g}$ is the same as the equation of elastostatic equilibrium with a vanishing shear stress:

$$\boldsymbol{\nabla} \cdot \mathbf{T} = \boldsymbol{\nabla} P = \rho \boldsymbol{g} = -\rho \boldsymbol{\nabla} \Phi \tag{12.3}$$

[Eq. (10.34) with $\mathbf{f} = -\boldsymbol{\nabla} \cdot \mathbf{T}$]. Here $\boldsymbol{g}$ is the acceleration of gravity (which need not be constant, e.g. it varies from location to location inside the Sun), and Φ is Newton's gravitational potential with

$$\boldsymbol{g} = -\boldsymbol{\nabla} \Phi . \tag{12.4}$$

Note our sign convention: Φ is negative near a gravitating body and zero far from all bodies.

From Eq. (12.3), we can draw some immediate and important inferences. Take the curl of Eq. (12.3):

$$\boldsymbol{\nabla} \Phi \times \boldsymbol{\nabla} \rho = 0 . \tag{12.5}$$

This tells us that, in hydrostatic equilibrium, the contours of constant density, coincide with the equipotential surfaces, i.e. $\rho = \rho(\Phi)$ and Eq. (eq:dbc) itself tells us that as we move from point to point in the fluid, the changes in P and Φ are related by $dP/d\Phi = -\rho(\Phi)$. This, in turn, implies that the difference in pressure between two equipotential surfaces Φ_1 and Φ_2 is given by

$$\Delta P = -\int_{\Phi_1}^{\Phi_2} \rho(\Phi) d\Phi, \tag{12.6}$$

Box 12.1
Thermodynamic Considerations

One feature of fluid dynamics, especially gas dynamics, that distinguishes it from elastodynamics, is that the thermodynamic properties of the fluid are often very important and we must treat energy conservation explicitly. In this box we review, from Chap. 4, some of the thermodynamic concepts we shall need in our study of fluids; see also, e.g., Reif (1959). We shall have no need for partition functions, ensembles and other statistical aspects of thermodynamics. Instead, we shall only need elementary thermodynamics.

We begin with the nonrelativistic first law of thermodynamics (4.8) for a sample of fluid with energy E, entropy S, volume V, number N_I of molecules of species I, temperature T, pressure P, and chemical potential μ_I for species I:

$$dE = TdS - PdV + \sum_I \mu_I dN_I . \tag{1}$$

Almost everywhere in our treatment of fluid mechanics (and throughout this chapter), we shall *assume that the term* $\sum_I \mu_I dN_I$ *vanishes.* Physically this happens because all relevant nuclear reactions are frozen (occur on timescles τ_{react} far longer than the dynamical timescales τ_{dyn} of interest to us), so $dN_I = 0$; and each chemical reaction is either frozen, or goes so rapidly ($\tau_{\text{react}} \ll \tau_{\text{dyn}}$) that it and its inverse are in local thermodynamic equilibrium (LTE): $\sum_I \mu_I dN_I = 0$ for those species involved in the reactions. In the intermediate situation, where some relevant reaction has $\tau_{\text{react}} \sim \tau_{\text{dyn}}$, we would have to carefully keep track of the relative abundances of the chemical or nuclear species and their chemical potentials.

Consider a small fluid element with mass Δm, energy per unit mass u, entropy per unit mass s, and volume per unit mass $1/\rho$. Then inserting $E = u\Delta m$, $S = s\Delta m$ and $V = \Delta m/\rho$ into the first law $dE = TdS - pdV$, we obtain the form of the first law that we shall use in almost all of our fluid dynamics studies:

$$du = Tds - Pd\left(\frac{1}{\rho}\right). \tag{2}$$

The internal energy (per unit mass) u comprises the random translational energy of the molecules that make up the fluid, together with the energy associated with their internal degrees of freedom (rotation, vibration etc.) and with their intermolecular forces. The term Tds represents some amount of heat (per unit mass) that may get injected into a fluid element, e.g. by viscous heating (last section of this chapter), or may get removed, e.g. by radiative cooling.

In fluid mechanics it is useful to introduce the enthalpy $H = E + PV$ of a fluid element (cf. Ex. 4.3) and the corresponding enthalpy per unit mass $h = u + P/\rho$. Inserting $u = h - P/\rho$ into the left side of the first law (1), we obtain the first law in the "enthalpy representation" [Eq. (4.23)]:

Box 12.1, Continued

$$dh = Tds + \frac{dP}{\rho}. \tag{3}$$

Because all reactions are frozen or are in LTE, the relative abundances of the various nuclear and chemical species are fully determined by a fluid element's density ρ and temperature T (or by any two other variables in the set ρ, T, s, and P). Correspondingly, the thermodynamic state of a fluid element is completely determined by any two of these variables. In order to calculate all features of that state from two variables, we must know the relevant *equations of state*, such as $P(\rho, T)$ and $s(\rho, T)$, or the fluid's fundamental thermodynamic potential (Table 4.1) from which follow the equations of state.

We shall often deal with *ideal gases* (in which intermolecular forces and the volume occupied by the molecules are both negligible). For any ideal gas, the equation of state $P(\rho, T)$ is

$$P = \frac{\rho k T}{\mu m_p} , \tag{4}$$

where μ is the *mean molecular weight* and m_p is the proton mass [cf. Eq. (3.57c) with the number density of particles $\bar{N}/V$ reexpressed as $\rho/\mu m_p$] . The mean molecular weight μ is the mean mass per gas molecule in units of the proton mass, and should not be confused with the chemical potential of species I, μ_I (which will rarely if ever be used in our fluid mechanics analyses).

An idealisation that is often accurate in fluid dynamics is that the fluid is *adiabatic*; that is to say there is no heating resulting from dissipative processes, such as viscosity, thermal conductivity or the emission and absorption of radiation. When this is a good approximation, the entropy per unit mass s of a fluid element is constant following a volume element with the flow, i.e.

$$\frac{ds}{dt} = 0. \tag{5}$$

In an adiabatic flow, there is only one thermodynamic degree of freedom and so we can write $P = P(\rho, s) = P(\rho)$. Of course, this function will be different for fluid elements that have different s. In the case of an ideal gas, a standard thermodynamic argument [Ex. 12.2] shows that the pressure in an adiabatically expanding or contracting fluid element varies with density as

$$P \propto \rho^\gamma , \tag{6}$$

where γ, the *adiabatic index*, is equal to the ratio of specific heats

$$\gamma = C_P/C_V . \tag{7}$$

Box 12.1, Continued

[Our specific heats, like the energy, entropy and enthalpy, are defined on a per unit mass basis, so $C_P = T(\partial s/\partial T)_P$ is the amount of heat that must be added to a unit mass of the fluid to increase its temperature by one unit, and similarly for $C_V = T(\partial s/\partial T)_\rho$.] A special case of adiabatic flow is *isentropic* flow. In this case, the entropy is constant everywhere, not just along individual streamlines.

Whenever the pressure can be regarded as a function of the density alone (the same function everywhere), the fluid is called *barotropic*. A particular type of barytrope is the *polytrope* in which $P \propto \rho^{1+1/n}$ for some constant n (the *polytropic index*. Another is a liquid of infinite bulk modulus for which ρ =constant, everywhere. Note that barytropes are not necessarily isentropes; for example, in a fluid of sufficiently high thermal conductivity, the temperature will be constant everywhere, thereby causing both P and s to be unique functions of ρ.

Moreover, as $\boldsymbol{\nabla} P \propto \boldsymbol{\nabla} \Phi$, the surfaces of constant pressure (the *isobars*) coincide with the gravitational equipotentials. This is all true when $\boldsymbol{g}$ varies inside the fluid, or when it is constant.

The gravitational acceleration $\boldsymbol{g}$ is actually constant to high accuracy in most non-astrophysical applications of fluid dynamics, for example on the surface of the earth. In this case, the pressure at a point in a fluid is, from Eq. (12.6), equal to the total weight of fluid per unit area above the point,

$$P(z) = g \int_z^\infty \rho dz \, , \tag{12.7}$$

where the integral is performed by integrating upward in the gravitational field; cf. Fig. 12.1). For example, the deepest point in the world's oceans is the bottom of the Marianas trench in the Pacific, 11.03 km. Adopting a density $\sim 10^3$kg m^{-3}for water and a value ~ 10 m s^{-2} for g, we obtain a pressure of $\sim 10^8$N m^{-2} or $\sim 10^3$ atmospheres. This is comparable with the yield stress of the strongest materials. It should therefore come as no surprize to discover that the deepest dive ever recorded by a submersible was made by the *Trieste* in 1960, when it reached a depth of 10.91 km, just a bit shy of the lowest point in the trench.

12.3.1 Archimedes' Law

The Law of Archimedes, states that when a solid body is totally or partially immersed in a liquid in a uniform gravitational field $\boldsymbol{g} = -g\mathbf{e}_z$, the total buoyant upward force of the liquid on the body is equal to the weight of the displaced liquid. A formal proof can be made as follows; see Fig. 12.2. The fluid, pressing inward on the body across a small element of the body's surface $d\Sigma$, exerts a force $d\mathbf{F}^{\text{buoy}} = \mathbf{T}(\_, -d\boldsymbol{\Sigma})$, where the minus sign is because, by convention, $d\boldsymbol{\Sigma}$ points out of the body rather than into it. Converting to index notation and integrating over the body's surface ∂V, we obtain for the net buoyant force

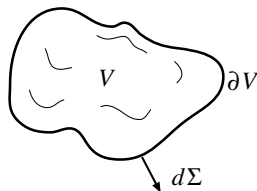$$F_i^{\text{buoy}} = -\int_{\partial V} T_{ij} d\Sigma_j \, . \tag{12.8}$$

Fig. 12.2: Derivation of Archimedes Law.

Now, imagine removing the body and replacing it by fluid that has the same pressure $P(z)$ and density $\rho(z)$, at each height z, as the surrounding fluid; this is the fluid that was originally displaced by the body. Since the fluid stress on ∂V has not changed, the buoyant force will be unchanged. Use Gauss's law to convert the surface integral (12.8) into a volume integral over the interior fluid (the originally displaced fluid)

$$F_i^{\text{buoy}} = - \int_V T_{ij;j} dV \ . \tag{12.9}$$

The displaced fluid obviously is in hydrostatic equilibrium with the surrounding fluid, and its equation of hydrostatic equilibrium (12.3), when inserted into Eq. (12.9), implies that

$$F_i^{\text{buoy}} = -\boldsymbol{g} \int_V \rho dV = -M\boldsymbol{g} \ , \tag{12.10}$$

where M is the mass of the displaced fluid. Thus, the upward buoyant force on the original body is equal in magnitude to the weight Mg of the displaced fluid. Clearly, if the body has a higher density than the fluid, then the downward gravitational force on it (its weight) will exceed the weight of the displaced fluid and thus exceed the buoyant force it feels, and the body will fall. If the body's density is less than that of the fluid, the buoyant force will exceed its weight and it will be pushed upward.

A key piece of physics underlying Archimedes law is the fact that the intermolecular forces acting in a fluid, like those in a solid (cf. Sec. 10.4), are of short range. If, instead, the forces were of long range, Archimedes' law could fail. For example, consider a fluid that is electrically conducting, with currents flowing through it that produce a magnetic field and resulting long-range magnetic forces (the magnetohydrodynamic situation studied in Chap. 18). If we then substitute an insulating solid for some region V of the conducting fluid, the force that acts on the solid will be different from the force that acted on the displaced fluid.

12.3.2 Stars and Planets

Stars and planets—if we ignore their rotation—are self-gravitating spheres, part fluid and part solid. We can model the structure of a such non-rotating, spherical, self-gravitating fluid body by combining the equation of hydrostatic equilibrium (12.3) in spherical polar coordinates,

$$\frac{dP}{dr} = -\rho \frac{d\Phi}{dr} \ , \tag{12.11}$$

with Poisson's equation,

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G\rho \,, \tag{12.12}$$

to obtain

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G\rho. \tag{12.13}$$

This can be integrated once radially with the aid of the boundary condition $dP/dr = 0$ at $r = 0$ (pressure cannot have a cusp-like singularity) to obtain

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2} \,, \tag{12.14a}$$

where

$$m = m(r) \equiv \int_0^r 4\pi \rho r^2 dr \tag{12.14b}$$

is the total mass inside radius r. Equation (12.14a) is an alternative form of the equation of hydrostatic equilibrium at radius r inside the body: Gm/r^2 is the gravitational acceleration g at r, $\rho Gm/r^2$ is the downward gravitational force per unit volume on the body's fluid, and dP/dr is the upward buoyant force per unit volume.

Equations (12.11)—(12.14b) are a good approximation for solid planets, as well as for stars and liquid planets, because, at the enormous stresses encountered in the interior of a solid planet, the strains are so large that plastic flow will occur. In other words, the limiting shear stresses are much smaller than the isotropic part of the stress tensor.

Let us make an order of magnitude estimate of the interior pressure in a star or planet of mass M and radius R. We use the equation of hydrostatic equilibrium (12.3) or (12.14a), approximating m by M, the density ρ by M/R^3 and the gravitational acceleration by GM/R^2, so that

$$P \sim \frac{GM^2}{R^4} \,. \tag{12.15}$$

In order to improve upon this estimate, we must solve Eq. (12.13). We therefore need a prescription for relating the pressure to the density. A common idealization is the polytropic relation, namely that

$$P \propto \rho^{1+1/n} \tag{12.16}$$

where n is called the polytropic index (cf. last part of Box 12.1). [This finesses the issue of the thermal balance of stellar interiors, which determines the temperature $T(r)$ and thence the pressure $P(\rho, T)$.] Low mass white dwarf stars are well approximated as $n = 1.5$ polytropes, and red giant stars are somewhat similar in structure to $n = 3$ polytropes. The giant planets, Jupiter and Saturn mainly comprise a H-He fluid which is well approximated by an $n = 1$ polytrope, and the density of a small planet like Mercury is very roughly constant $(n = 0)$. We also need boundary conditions to solve Eqs. (12.14). We can choose some density ρ_c and corresponding pressure $P_c = P(\rho_c)$ at the star's center $r = 0$, then integrate Eqs. (12.14) outward until the pressure P drops to zero, which will be the star's (or planet's) surface. The values of r and m there will the the star's radius R and mass M. For details of polytropic stellar models constructed in this manner see, e.g., Chandrasekhar (1939); for the case $n = 2$, see Ex. 12.5 below.

We can easily solve the equation of hydrostatic equilibrium (12.14a) for a constant density $(n = 0)$ star to obtain

$$P = P_0 \left(1 - \frac{r^2}{R^2} \right) , \tag{12.17}$$

where the central pressure is

$$P_0 = \left(\frac{3}{8\pi} \right) \frac{GM^2}{R^4} , \tag{12.18}$$

consistent with our order of magnitude estimate (12.15).

12.3.3 Hydrostatics of Rotating Fluids

The equation of hydrostatic equilbrium (12.3) and the applications of it discussed above are valid only when the fluid is static in a reference frame that is rotationally inertial. However, they are readily extended to bodies that rotate rigidly, with some uniform angular velocity $\boldsymbol{\Omega}$ relative to an inertial frame. In a frame that corotates with the body, the fluid will have vanishing velocity $\mathbf{v}$, i.e. will be static, and the equation of hydrostatic equilibrium (12.3) will be changed only by the addition of the centrifugal force per unit volume:

$$\boldsymbol{\nabla} P = \rho(\boldsymbol{g} + \mathbf{g}_{\text{cen}}) = -\rho \boldsymbol{\nabla} (\Phi + \Phi_{\text{cen}}) . \tag{12.19}$$

Here

$$\mathbf{g}_{\text{cen}} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\boldsymbol{\nabla} \Phi_{\text{cen}} , \tag{12.20}$$

is the centrifugal acceleration, $\rho \boldsymbol{g}_{\text{cen}}$ is the centrifugal force per unit volume, and

$$\Phi_{\text{cen}} = -\frac{1}{2} (\boldsymbol{\Omega} \times \mathbf{r})^2. \tag{12.21}$$

is a *centrifugal potential* whose gradient is equal to the centrifugal acceleration in our situation of constant $\boldsymbol{\Omega}$. The centrifugal potential can be regarded as an augmentation of the gravitational potential Φ. Indeed, *in the presence of uniform rotation, all hydrostatic theorems [e.g., Eqs. (12.5) and (12.6)] remain valid with Φ replaced by $\Phi + \Phi_{\text{cen}}$.*

We can illustrate this by considering the shape of a spinning fluid planet. Let us suppose that almost all the mass of the planet is concentrated in its core so that the gravitational potential $\Phi = -GM/r$ is unaffected by the rotation. (Here M is the planet mass and r is the radius.) Now the surface of the planet must be an equipotential of $\Phi + \Phi_{\text{cen}}$ (coinciding with the zero-pressure isobar) [cf. Eq. (12.5) and subsequent sentences, with $\Phi \to \Phi + \Phi_{\text{cen}}$]. The contribution of the centrifugal potential at the equator is $-\Omega^2 R_e^2/2$ and at the pole zero. The difference in the gravitational potential Φ between the equator and the pole is $\sim g(R_e - R_p)$ where R_e, R_p are the equatorial and polar radii respectively and g is the gravitational acceleration at the planet's surface. Therefore, adopting this centralized-mass model, we estimate the difference between the polar and equatorial radii to be

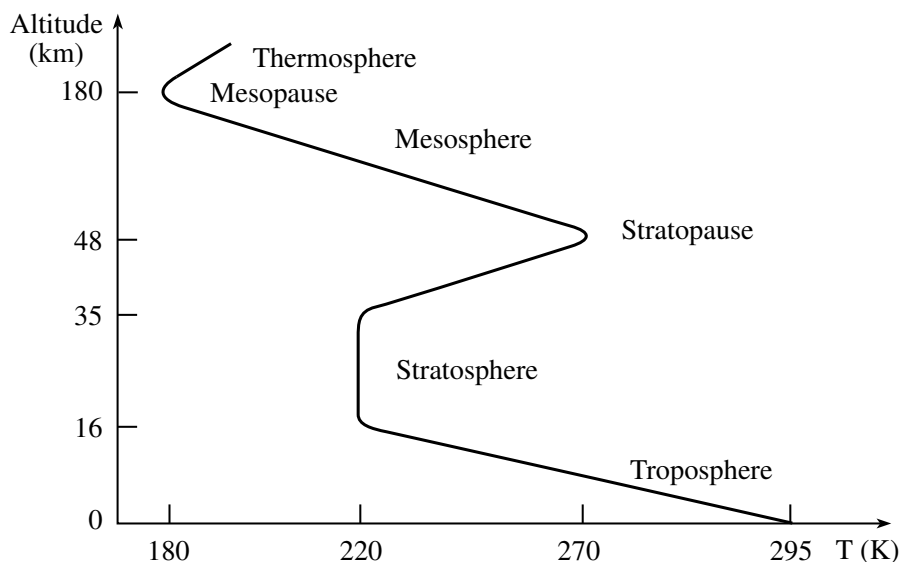$$R_e - R_p \simeq \frac{\Omega^2 R^2}{2g} \tag{12.22}$$

Fig. 12.3: Actual temperature variation in the Earth's mean atmosphere at temperate latitudes.

The earth, although not a fluid, is unable to withstand large shear stresses (because its shear strain cannot exceed ~ 0.001); therefore its surface will not deviate by more than the maximum height of a mountain from its equipotential. If we substitute $g \sim 10 \mathrm{m\ s^{-2}}$, $R \sim 6 \times 10^6 \mathrm{m}$ and $\Omega \sim 7 \times 10^{-5} \mathrm{rad\ s^{-1}}$, we obtain $R_e - R_p \sim 10\mathrm{km}$, about half the correct value of 21km. The reason for this discrepancy lies in our assumption that all the mass lies in the center. In fact it is distributed fairly uniformly in radius and, in particular, some mass is found in the equatorial bulge. This deforms the gravitational equipotential surfaces from spheres to ellipsoids, which accentuates the flattening. If, following Newton (in his *Principia Mathematica* 1687), we assume that the earth has uniform density then the flattening estimate is about 2.5 times larger than the actual flattening (Ex. 12.6).

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EXERCISES

Exercise 12.1 *Practice: Weight in Vacuum*
How much more would you weigh *in vacuo*?

Exercise 12.2 *Derivation: Adiabatic Index*
Show that for an ideal gas [one with equation of state $P = (k/\mu m_p)\rho T$; Eq. (4) of Box 12.1], the specific heats are related by $C_P = C_V + k/(\mu m_p)$, and the adiabatic index is $\gamma = C_P/C_V$. [The solution is given in most thermodynamics textbooks.]

Exercise 12.3 *Example: Earth's Atmosphere*
As mountaineers know, it gets cooler as you climb. However, the rate at which the temperature falls with altitude depends upon the assumed thermal properties of air. Consider two limiting cases.
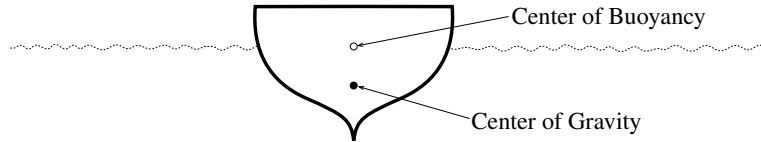
Fig. 12.4: Stability of a Boat. We can understand the stability of a boat to small rolling motions by defining both a center of gravity for weight of the boat and also a center of buoyancy for the upthrust exerted by the water.

(i) In the lower stratosphere, the air is isothermal. Use the equation of hydrostatic equilibrium (12.3) to show that the pressure decreases exponentially with height z

$$P \propto \exp(-z/H),$$

where the scale height H is given by

$$H = \frac{kT}{\mu m_p g}$$

and μ is the mean molecular weight of air and m_p is the proton mass. Estimate your local isothermal scale height.

(ii) Suppose that the air is isentropic so that $P \propto \rho^\gamma$, where γ is the specific heat ratio. (For diatomic gases like nitrogen and oxygen, $\gamma \sim 1.4$.) Show that the temperature gradient satisfies

$$\frac{dT}{dz} = -\frac{\gamma - 1}{\gamma} \frac{g\mu m_p}{k}.$$

Note that the temperature gradient vanishes when $\gamma \to 1$. Evaluate the temperature gradient, otherwise known as the *lapse rate*. At low altitude, the average lapse rate is measured to be $\sim 6\text{K km}^{-1}$, Show that this is intermediate between the two limiting cases (Figure 12.3).

Exercise 12.4 *Problem: Stability of Boats*
Use Archimedes Principle to explain qualitatively the conditions under which a boat floating in still water will be stable to small rolling motions from side to side. (Hint, you might want to introduce a *center of buoyancy* inside the boat, as in Figure 12.4.

Exercise 12.5 *Problem: Jupiter and Saturn*
The text described how to compute the central pressure of a non-rotating, constant density planet. Repeat this exercise for the polytropic relation $P = K\rho^2$ (polytropic index $n = 1$), appropriate to Jupiter and Saturn. Use the information that $M_J = 2 \times 10^{27}\text{kg}$, $M_S = 6 \times 10^{26}\text{kg}$, $R_J = 7 \times 10^4\text{km}$ to estimate the radius of Saturn. Hence, compute the central pressures, gravitational binding energy and polar moments of inertia of both planets.

Exercise 12.6 *Example: Shape of a constant density, spinning planet*

(i) Show that the spatially variable part of the gravitational potential for a uniform density, non-rotating planet can be written as $\Phi = 2\pi G\rho r^2/3$, where ρ is the density.

(ii) Hence argue that the gravitational potential for a slowly spinning planet can be written in the form

$$\Phi = \frac{2\pi G\rho r^2}{3} + Ar^2 P_2(\mu)$$

where A is a constant and P_2 is a Legendre polynomial of $\mu = \sin(\text{latitude})$. What happens to the P_1 term?

(iii) Give an equivalent expansion for the potential outside the planet.

(iv) Now transform into a frame spinning with the planet and add the centrifugal potential to give a total potential.

(v) By equating the potential and its gradient at the planet's surface, show that the difference between the polar and the equatorial radii is given by

$$R_e - R_p \simeq \frac{5\Omega^2 R^2}{2g},$$

where g is the surface gravity. Note that this is 5 times the answer for a planet whose mass is all concentrated at its center [Eq. (12.22)].

Exercise 12.7 *Problem: Shapes of Stars in a Tidally Locked Binary System*
Consider two stars, with the same mass M orbiting each other in a circular orbit with diameter (separation between the stars' centers) a. Kepler's laws tell us that their orbital angular velocity is $\Omega = \sqrt{2M/a^3}$. Assume that each star's mass is concentrated near its center so that everywhere except near a star's center the gravitational potential, in an inertial frame, is $\Phi = -GM/r_1 - GM/r_2$ with r_1 and r_2 the distances of the observation point from the center of star 1 and star 2. Suppose that the two stars are "tidally locked", i.e. tidal gravitational forces have driven them each to rotate with rotational angular velocity equal to the orbital angular velocity Ω. (The moon is tidally locked to the earth; that is why it always keeps the same face toward the earth.) Then in a reference frame that rotates with angular velocity Ω, each star's gas will be at rest, $\mathbf{v} = 0$.

(a) Write down the total potential $\Phi + \Phi_{\text{cen}}$ for this binary system.

(b) Using Mathematica or Maple or some other computer software, plot the equipotentials $\Phi + \Phi_{\text{cen}} = (\text{constant})$ for this binary in its orbital plane, and use these equipotentials to describe the shapes that these stars will take if they expand to larger and larger radii (with a and M held fixed). You should obtain a sequence in which the stars, when compact, are well separated and nearly round, and as they grow tidal gravity elongates them, ultimately into tear-drop shapes followed by merger into a single, highly distorted star. With further expansion there should come a point where they start flinging mass off into the surrounding space (a process not included in this hydrostatic analysis).

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12.4   Conservation Laws

As a foundation for making the transition from hydrostatics to hydrodynamics [to situations with nonzero fluid velocity $\mathbf{v}(\mathbf{x}, t)$], we shall give a general discussion of conservation laws, focusing especially on the conservation of mass and of linear momentum. Our discussion will be the Newtonian version of the special relativistic ideas we developed in Sec. 1.12.

   We have already met and briefly used the law of Newtonian mass conservation $\partial \rho / \partial t + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$ in the theory of elastodynamics, Eq. (11.2c). However, it is worthwhile to pause now and pay closer attention to how this law of mass conservation arises.

   Consider a continuous substance (not necessarily a fluid) with mass density $\rho(\mathbf{x}, t)$, and a small elementary volume $\mathcal{V}$, fixed in space (i.e., fixed in some Newtonian reference frame). If the matter moves, then there will be a flow of mass, a *mass flux* across each element of surface $d\boldsymbol{\Sigma}$ on the boundary $\partial \mathcal{V}$ of $\mathcal{V}$. Provided that there are no sources or sinks of matter, the total rate of change of the mass residing within $\mathcal{V}$ will be given by the net rate of mass transport across $\partial \mathcal{V}$. Now the rate at which mass moves across a unit area is the *mass flux*, $\rho \mathbf{v}$, where $\mathbf{v}(\mathbf{x}, t)$ is the velocity field. We can therefore write

$$\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho dV = - \int_{\partial \mathcal{V}} \rho \mathbf{v} \cdot d\boldsymbol{\Sigma}. \tag{12.23}$$

(Remember that the surface is fixed in space.) Invoking Gauss' theorem, we obtain

$$\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho dV = - \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) dV. \tag{12.24}$$

As this must be true for arbitrary small volumes $\mathcal{V}$, we can abstract the differential equation of mass conservation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0. \tag{12.25}$$

This is the Newtonian analog of the relativistic discussion of conservation laws in Sec. 1.11 and Fig. 1.14.

   Writing the conservation equation in the form (12.25), where we monitor the changing density at a given location in space, rather than moving with the material, is called the *Eulerian* approach. There is an alternative *Lagrangian* approach to mass conservation, in which we focus on changes of density as measured by somebody who moves, locally, with the material, i.e. with velocity $\mathbf{v}$. We obtain this approach by differentiating the product $\rho \mathbf{v}$ in Eq. (12.25), to obtain

$$\frac{d\rho}{dt} = -\rho \cdot \mathbf{v} \, , \tag{12.26}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \, . \tag{12.27}$$

The operator d/dt is known as the *convective time derivative* (or *advective time derivative*) and crops up often in continuum mechanics. Its physical interpretation is very simple. Consider first the partial derivative $(\partial/\partial t)_\mathbf{x}$. This is the rate of change of some quantity [the density ρ in Eq. (12.26)] at a fixed point in space in some reference frame. In other words, if there is motion, $\partial/\partial t$ compares this quantity at the same point in space for two different points in the material. By contrast, the convective time derivative, (d/dt) follows the motion and takes the difference in the value of the quantity at successive instants at the same point in the moving matter. It therefore measures the rate of change of the physical quantity following the material rather than at a fixed point in space; it is the time derivative for the Lagrangian approach. Note that the convective derivative is the Newtonian limit of relativity's proper time derivative along the world line of a bit of matter, $d/d\tau = u^\alpha \partial/\partial x^\alpha = (dx^\alpha/d\tau)\partial/\partial x^\alpha$ [Secs. 1.4 and 1.6].

Equation (12.25) is our model for Newtonian conservation laws. It says that there is a quantity, in this case *mass*, with a certain density, in this case ρ, and a certain flux, in this case $\rho\mathbf{v}$, and this quantity is neither created nor destroyed. The temporal derivative of the density (at a fixed point in space) added to the divergence of the flux must then vanish. Of course, not all physical quantities have to be conserved. If there were sources or sinks of mass, then these would be added to the right hand side of Eq. (12.25).

Turn, now, to momentum conservation. Of course, we have met momentum conservation previously, in relativity quite generally (Sec. 1.12), and in the Newtonian treatment of elasticity theory (Sec. 11.2.1 However it is useful, as we embark on fluid mechanics where momentum conservation can become rather complex, to examine its Newtonian foundations.

If we just consider the *mechanical momentum* associated with the motion of mass, its density is the vector field $\rho\mathbf{v}$. There can also be other forms of momentum density, e.g. electromagnetic, but these do not enter into fluid mechanics; for fluids we need only consider $\rho\mathbf{v}$.

The momentum flux is more interesting and rich: The mechanical momentum $d\mathbf{p}$ crossing a small element of area $d\mathbf{\Sigma}$, from the back side of $d\mathbf{\Sigma}$ to the front (in the "positive sense"; cf. Fig. 1.13b) during unit time is given by $d\mathbf{p} = (\rho\mathbf{v} \cdot d\mathbf{\Sigma})\mathbf{v}$. This is also a vector and it is a linear function of the element of area $d\mathbf{\Sigma}$. This then allows us to define a second rank tensor, the *mechanical momentum flux* (i.e. the momentum flux carried by the moving mass), by the equivalent relations

$$d\mathbf{p} = (\rho\mathbf{v} \cdot d\mathbf{\Sigma})\mathbf{v} = \mathbf{T}_m(\_\_, d\mathbf{\Sigma}) , \quad \mathbf{T}_m = \rho\mathbf{v} \otimes \mathbf{v}. \tag{12.28}$$

This tensor is manifestly symmetric; it is the quantity that we alluded to in footnote 1 of Chap. 11 (elastocynamics) and then ignored.

We are now in a position to write down a conservation law for momentum by direct analogy with Eq. (12.25), namely

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \mathbf{\nabla} \cdot \mathbf{T}_m = \mathbf{f}, \tag{12.29}$$

where $\mathbf{f}$ is the net rate of increase of momentum in a unit volume due to all forces that act on the material, i.e. the force per unit volume.

In elasticity theory we showed that the elastic force per unit volume could be written as the divergence of an elastic stress tensor, $\mathbf{f}_{el} = -\boldsymbol{\nabla} \cdot \mathbf{T}_{el}$ there by permitting us to put momentum conservation into the standard conservation-law form $\partial(\rho\mathbf{v})/\partial t + \boldsymbol{\nabla} \cdot \mathbf{T}_{el} = 0$. In order to be able to go back and forth between a differential conservation law and an integral conservation law, as we did in the case of rest mass [Eqs. (12.23)–(12.25)], it is necessary that the differential conservation law take the form "time derivative of density of something, plus divergence of the flux of that something, vanishes". Accordingly, in the differential conservation law for momentum (12.29), it must always be true — regardless of the physical nature of the force $\mathbf{f}$ — that there is a stress tensor $\mathbf{T}_f$ such that

$$\mathbf{f} = -\boldsymbol{\nabla} \cdot \mathbf{T}_f \,, \tag{12.30}$$

so Eq. (12.29) becomes

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{T}_m + \mathbf{T}_f) = 0 \,. \tag{12.31}$$

We have used the symbol $\mathbf{T}_m$ for the mechanical momentum flux because, as well as being the momentum crossing unit area in unit time, it is also a piece of the stress tensor (force per unit area): the force that acts across a unit area is just the rate at which momentum crosses that area; cf. the relativistic discussion in Sec. 1.12. Correspondingly, the total stress tensor for the material is the sum of the mechanical piece and the force piece,

$$\mathbf{T} = \mathbf{T}_m + \mathbf{T}_f \,. \tag{12.32}$$

and the momentum conservation law says, simply,

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T} = 0 \,. \tag{12.33}$$

Evidently, a knowledge of the stress tensor $\mathbf{T}$ for some material is equivalent to a knowledge of the force density $\mathbf{f}$ that acts on it. Now, it often turns out to be much easier to figure out the form of the stress tensor, for a given situation, than the form of the force. Correspondingly, as we add new pieces of physics to our fluid analysis (isotropic pressure, viscosity, gravity, magnetic forces), an efficient way to proceed at each stage is to insert the relevant physics into the stress tensor $\mathbf{T}$, and then evaluate the resulting contribution $\mathbf{f} = -\boldsymbol{\nabla} \cdot \mathbf{T}$ to the force and thence to the momentum conservation law (12.33). At each step, *we get out in $\mathbf{f} = -\boldsymbol{\nabla} \cdot \mathbf{T}$ the physics that we put into $\mathbf{T}$*.

We can proceed in the same way with energy conservation. There is an energy density $U(\mathbf{x}, t)$ for a fluid and an energy flux $\mathbf{F}(\mathbf{x}, t)$, and they obey a conservation law with the standard form

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{F} = 0 \,. \tag{12.34}$$

At each stage in our buildup of fluid mechanics (adding, one by one, the influences of compressional energy, viscosity, gravity, magnetism), we can identify the relevant contributions to U and $\mathbf{F}$ and then grind out the resulting conservation law (12.34). At each stage we get out the physics that we put into U and $\mathbf{F}$.

We conclude this section with two remarks. The first is that in going from Newtonian physics (this chapter) to special relativity (Chap. 1), mass and energy get combined (added)

to form a conserved mass-energy or total energy; that total energy and the momentum are the temporal and spatial parts of a spacetime 4-vector, the 4-momentum; and correspondingly, the conservation laws for mass [Eq. (12.25)], nonrelativistic energy [Eq. (12.34)], and momentum [Eq. (12.33)] get unified into a single conservation law for 4-momentum, which is expressed as the vanishing 4-dimensional, spacetime divergence of the 4-dimensional stress-energy tensor (Sec. 1.12). The second remark is that there may seem something tautological about our procedure. We have argued that the mechanical momentum will not be conserved in the presence of forces such as elastic forces. Then we argued that we can actually associate a momentum flux, or more properly a stress tensor, with the strained elastic medium, so that the combined momentum is conserved. It is almost as if we regard conservation of momentum as a principle to be preserved at all costs and so every time there appears to be a momentum deficit, we simply define it as a bit of the momentum flux. (An analogous accusation could be made about the conservation of energy.) This, however, is not the whole story. What is important is that the force density $\mathbf{f}$ can always be expressed as the divergence of a stress tensor; that fact is central to the nature of force and of momentum conservation. An erroneous formulation of the force would not necessarily have this property and there would not be a differential conservation law. So the fact that we *can* create elastostatic, thermodynamic, viscous, electromagnetic, gravitational *etc* contributions to some grand stress tensor (that go to zero outside the regions occupied by the relevant matter or fields), as we shall see in the coming chapters, *is* significant and affirms that our physical model is complete at the level of approximation to which we are working.

12.5   Conservation Laws for an Ideal Fluid

We now turn from hydrostatic situations to fully dynamical fluids. We shall derive the fundamental equations of fluid dynamics in several stages. In this section, we will confine our attention to *ideal* fluids, i.e., flows for which it is safe to ignore dissipative processes (viscosity and thermal conductivity), and for which, therefore, the entropy of a fluid element remains constant with time. In the next section we will introduce the effects of viscosity, and in Chap. 17 we will introduce heat conductivity. At each stage, we will derive the fundamental fluid equations from the even-more-fundamental conservation laws for mass, momentum, and energy.

12.5.1   Mass Conservation

Mass conservation, as we have seen, takes the (Eulerian) form $\partial \rho / \partial t + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$ [Eq. (12.25)], or equivalently the (Lagrangian) form $d\rho/dt = -\rho \boldsymbol{\nabla} \cdot \mathbf{v}$ [Eq. (12.26)], where $d/dt = \partial/\partial t + \mathbf{v} \cdot \boldsymbol{\nabla}$ is the convective time derivative (moving with the fluid) [Eq. (12.27)].

We define a fluid to be *incompressible* when $d\rho/dt = 0$. *Note:* incompressibility does *not* mean that the fluid cannot be compressed; rather, it merely means that in the situation being studied, the density of each fluid element remains constant as time passes. From Eq. (12.27), we see that incompressibility implies that the velocity field has vanishing divergence (i.e. it is *solenoidal*, i.e. expressible as the curl of some potential). The condition that the fluid be incompressible is a weaker condition than that the density be constant everywhere;

for example, the density varies substantially from the earth's center to its surface, but if the material inside the earth were moving more or less on surfaces of constant radius, the flow would be incompressible. As we shall shortly see, approximating a flow as incompressible is a good approximation when the flow speed is much less than the speed of sound and the fluid does not move through too great gravitational potential differences.

12.5.2 Momentum Conservation

For an ideal fluid, the only forces that can act are those of gravity and of the fluid's isotropic pressure P. We have already met and discussed the contribution of P to the stress tensor, $\mathbf{T} = P\mathbf{g}$, when dealing with elastic media (Chap. 10) and in hydrostatics (Sec. 12.3 The gravitational force density, $\rho\boldsymbol{g}$, is so familiar that it is easier to write it down than the corresponding gravitational contribution to the stress. Correspondingly, we can most easily write momentum conservation in the form

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T} = \rho\boldsymbol{g} \; ; \quad \text{i.e.} \quad \frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{v} \otimes \mathbf{v} + P\mathbf{g}) = \rho\boldsymbol{g} \; , \tag{12.35}$$

where the stress tensor is given by

$$\mathbf{T} = \rho\mathbf{v} \otimes \mathbf{v} + P\mathbf{g}. \tag{12.36}$$

[cf. Eqs. (12.28), (12.29) and (12.3)]. The first term, $\rho\mathbf{v} \otimes \mathbf{v}$, is the mechanical momentum flux (also called the *kinetic* stress), and the second, $P\mathbf{g}$, is that associated with the fluid's pressure.

   In most of our applications, the gravitational field $\boldsymbol{g}$ will be externally imposed, i.e., it will be produced by some object such as the Earth that is different from the fluid we are studying. However, the law of momentum conservation remains the same, Eq. (12.35), independently of what produces gravity, the fluid or an external body or both. And independently of its source, one can write the stress tensor $\mathbf{T}_g$ for the gravitational field $\mathbf{g}$ in a form presented and discussed in Box 12.2 below — a form that has the required property $-\boldsymbol{\nabla} \cdot \mathbf{T}_g = \rho\mathbf{g} = $ (the gravitational force density).

12.5.3 Euler Equation

The "Euler equation" is the equation of motion that one gets out of the momentum conservation law (12.35) by performing the differentiations and invoking mass conservation (12.25):

$$\frac{d\mathbf{v}}{dt} = -\frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{g} \; . \tag{12.37}$$

This Euler equation was first derived in 1785 by the Swiss mathematician and physicist Leonhard Euler.

   The Euler equation has a very simple physical interpretation: $d\mathbf{v}/dt$ is the convective derivative of the velocity, i.e. the derivative moving with the fluid, which means it is the acceleration felt by the fluid. This acceleration has two causes: gravity, $\boldsymbol{g}$, and the pressure

gradient $\boldsymbol{\nabla} P$. In a hydrostatic situation, $\mathbf{v} = 0$, the Euler equation reduces to the equation of hydrostatic equilibrium, $\boldsymbol{\nabla} P = \rho \boldsymbol{g}$ [Eq. (12.5)]

In Cartesian coordinates, the Euler equation (12.37) and mass conservation (12.25) comprise four equations in five unknowns, ρ, P, v_x, v_y, v_z. In order to close this system of equations, we must relate P to ρ. For an ideal fluid, we use the fact that the entropy of each fluid element is conserved (because there is no mechanism for dissipation),

$$\frac{ds}{dt} = 0 \; , \tag{12.38}$$

together with an equation of state for the pressure in terms of the density and the entropy, $P = P(\rho, s)$. In practice, the equation of state is often well approximated by incompressibility, $\rho = $ constant, or by a polytropic relation, $P = K(s)\rho^{1+1/n}$ [Eq. (12.16)].

12.5.4 Bernoulli's Principle; Expansion, Vorticity and Shear

Bernoulli's principle is well known. Less well appreciated are the conditions under which it is true. In order to deduce these, we must first introduce a kinematic quantity known as the *vorticity*,

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}. \tag{12.39}$$

The attentive reader may have noticed that there is a parallel between elasticity and fluid dynamics. In elasticity, we are concerned with the gradient of the displacement vector field $\boldsymbol{\xi}$ and we decompose it into expansion, rotation and shear. In fluid dynamics, we are interested in the gradient of the velocity field $\mathbf{v} = d\boldsymbol{\xi}/dt$ and we make an analogous decomposition. The fluid analogue of expansion $\Theta = \boldsymbol{\nabla} \cdot \boldsymbol{\xi}$ is its time derivative $\theta \equiv \boldsymbol{\nabla} \cdot \mathbf{v} = d\Theta/dt$, which we call the *rate of expansion*. This has already appeared in the equation of mass conservation. Rotation $\boldsymbol{\phi} = \frac{1}{2}\boldsymbol{\nabla} \times \xi$ is uninteresting in elastostatics because it causes no stress. Vorticity $\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \mathbf{v} = 2d\boldsymbol{\phi}/dt$ is its fluid counterpart, and although primarily a kinematic quantity, it plays a vital role in fluid dynamics because of its close relation to angular momentum; we shall discuss it in more detail in the following chapter. Shear $\boldsymbol{\Sigma}$ is responsible for the shear stress in elasticity. We shall meet its counterpart, the *rate of shear* tensor $\boldsymbol{\sigma} = d\boldsymbol{\Sigma}/dt$ below when we introduce the viscous stress tensor.

To derive the Bernoulli principle, we begin with the Euler equation $d\mathbf{v}/dt = -(1/\rho)\boldsymbol{\nabla} P + \boldsymbol{g}$; we express $\boldsymbol{g}$ as $-\boldsymbol{\nabla}\Phi$; we convert the convective derivative of velocity (i.e. the acceleration) into its two parts $d\mathbf{v}/dt = \partial\mathbf{v}/\partial t + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$; and we rewrite $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$ using the vector identity

$$\mathbf{v} \times \boldsymbol{\omega} \equiv \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \frac{1}{2}\boldsymbol{\nabla} v^2 - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \; . \tag{12.40}$$

The result is

$$\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\nabla}(\frac{1}{2}v^2 + \Phi) + \frac{\boldsymbol{\nabla} P}{\rho} - \mathbf{v} \times \boldsymbol{\omega} = 0. \tag{12.41}$$

This is just the Euler equation written in a new form, but it is also the most general version of the Bernoulli principle. Two special cases are of interest:

(i) *Steady flow of an ideal fluid.* A steady flow is one in which $\partial(\text{everything})/\partial t = 0$, and an ideal fluid is one in which dissipation (due to viscosity and heat flow) can be ignored. Ideality implies that the entropy is constant following the flow, i.e. $ds/dt = (\mathbf{v}\cdot\boldsymbol{\nabla})s = 0$. From the thermodynamic identity, $dh = Tds + dP/\rho$ [Eq. (3) of Box 12.1] we obtain

$$(\mathbf{v} \cdot \boldsymbol{\nabla})P = \rho(\mathbf{v} \cdot \boldsymbol{\nabla})h. \qquad (12.42)$$

(Remember that the flow is steady so there are no time derivatives.) Now, define the *Bernoulli constant, B*, by

$$B \equiv \frac{1}{2}v^2 + h + \Phi . \qquad (12.43)$$

This allows us to take the scalar product of the gradient of Eq. (12.43) with the velocity $\mathbf{v}$ to rewrite Eq. (12.41) in the form

$$\frac{dB}{dt} = (\mathbf{v} \cdot \boldsymbol{\nabla})B = 0, \qquad (12.44)$$

This says that the Bernoulli constant, like the entropy, does not change with time in a fluid element. Let us define *streamlines*, analogous to lines of force of a magnetic field, by the differential equations

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \qquad (12.45)$$

In the language of Sec. 1.5, these are just the integral curves of the (steady) velocity field; they are also the spatial world lines of the fluid elements. Equation (12.44) says that *the Bernoulli constant is constant along streamlines in a steady, ideal flow.*

(ii) *Irrotational flow of an isentropic fluid.* An even more specialized type of flow is one where the vorticity vanishes and the entropy is constant *everywhere*. A flow in which $\boldsymbol{\omega} = 0$ is called an *irrotational* flow. (Later we shall learn that, if an incompressible flow initially is irrotational and it encounters no walls and experiences no significant viscous stresses, then it remains always irrotational.) Now, as the curl of the velocity field vanishes, we can follow the electrostatic precedent and introduce a *velocity potential* $\psi(\mathbf{x}, t)$ so that at any time,

$$\mathbf{v} = \boldsymbol{\nabla}\psi. \qquad (12.46)$$

A flow in which the entropy is constant everywhere is called *isentropic* (Box 12.1). In an isentropic flow, $\boldsymbol{\nabla}P = \rho\boldsymbol{\nabla}h$. Imposing these conditions on Eq. (12.41), we obtain, for an isentropic, irrotational flow:

$$\boldsymbol{\nabla}\left[\frac{\partial\psi}{\partial t} + B\right] = 0. \qquad (12.47)$$

Thus, the quantity $\partial\psi/\partial t + B$ will be constant everywhere in the flow, not just along streamlines. Of course, if the flow is steady so $\partial/\partial t(\text{everything}) = 0$, then B itself is constant. Note the important restriction that the vorticity in the flow vanish.
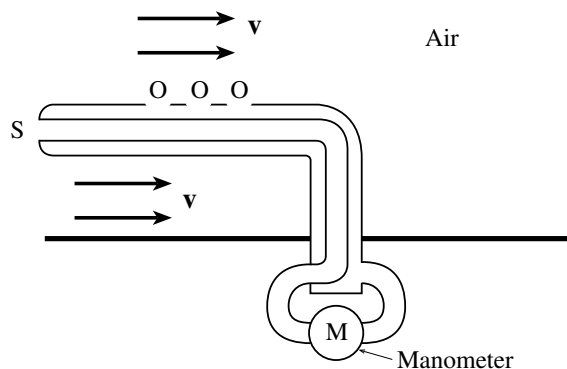
Fig. 12.5: Schematic illustration of a Pitot tube used to measure airspeed. The tube points into the flow well away from the boundary layer. A manometer measures the pressure difference between the stagnation points S, where the external velocity is very small, and several orifices O in the side of the tube where the pressure is almost equal to that in the free air flow. The air speed can then be inferred by application of the Bernoulli principle.

The most immediate consequence of Bernoulli's theorem in a steady, ideal flow (constancy of $B = \frac{1}{2}v^2 + h + \Phi$ along flow lines) is that the enthalpy falls when the speed increases. For a perfect gas the enthalpy is simply $h = c^2/(\gamma - 1)$, where c is the speed of sound. For an incompressible liquid, it is P/ρ. Microscopically, what is happening is that we can decompose the motion of the constituent molecules into a bulk motion and a random motion. The total kinetic energy should be constant after allowing for variation in the gravitational potential. As the bulk kinetic energy increases, the random or thermal kinetic energy must decrease, leading to a reduction in pressure.

A simple, though important application of the Bernoulli principle is to the Pitot tube which is used to measure air speed in an aircraft (Figure 12.5). A Pitot tube extends out from the side of the aircraft and points into the flow. There is one small orfice at the end where the speed of the gas relative to the tube is small and several apertures along the tube, where the gas moves with approximately the air speed. The pressure difference between the end of the tube and the sides is measured using an instrument called a manometer and is then converted into an airspeed using the formula $v = (2\Delta P/\rho)^{1/2}$. For $v \sim 100\text{m s}^{-1}, \rho \sim 1\text{kg m}^{-3}$, $\Delta P \sim 5000\text{N m}^{-3} \sim 0.05$atmospheres. Note that the density of the air ρ will vary with height.

12.5.5 Conservation of Energy

As well as imposing conservation of mass and momentum, we must also address energy conservation. So far, in our treatment of fluid dynamics, we have finessed this issue by simply postulating some relationship between the pressure P and the density ρ. In the case of ideal fluids, this is derived by requiring that the entropy be constant following the flow. In this case, we are not required to consider the energy to derive the flow. However, understanding how energy is conserved is often very useful for gaining physical insight. Furthermore, it is imperative when dissipative processes operate.

The most fundamental formulation of the law of energy conservation is Eq. (12.34):

Quantity	Density	Flux
Mass	ρ	$\rho\mathbf{v}$
Momentum	$\rho\mathbf{v}$	$\mathbf{T} = P\boldsymbol{g} + \rho\mathbf{v} \otimes \mathbf{v}$
Energy	$U = (\frac{1}{2}v^2 + u + \Phi)\rho$	$\mathbf{F} = (\frac{1}{2}v^2 + h + \Phi)\rho\mathbf{v}$

Table 12.1: Densities and Fluxes of mass, momentum, and energy for an ideal fluid in an externally produced gravitational field.

$\partial U/\partial t + \boldsymbol{\nabla} \cdot \mathbf{F} = 0$. To explore its consequences for an ideal fluid, we must insert the appropriate ideal-fluid forms of the energy density U and energy flux $\mathbf{F}$.

When (for simplicity) the fluid is in an externally produced gravitational field Φ, its energy density is obviously

$$U = \rho \left(\frac{1}{2}v^2 + u + \Phi \right) , \tag{12.48}$$

where the three terms are kinetic, internal, and gravitational. When the fluid participates in producing gravity and one includes the energy of the gravitational field itself, the energy density is a bit more subtle; see Box 12.2.

In an external field one might expect the energy flux to be $\mathbf{F} = U\mathbf{v}$, but this is not quite correct. Consider a bit of surface area dA orthogonal to the direction in which the fluid is moving, i.e., orthogonal to $\mathbf{v}$. The fluid element that crosses dA during time dt moves through a distance $dl = vdt$, and as it moves, the fluid behind this element exerts a force PdA on it. That force, acting through the distance dl, feeds an energy $dE = (PdA)dl = PvdAdt$ across dA; the corresponding energy flux across dA has magnitude $dE/dAdt = Pv$ and obviously points in the $\mathbf{v}$ direction, so it contributes $P\mathbf{v}$ to the energy flux $\mathbf{F}$. This contribution is missing from our initial guess $\mathbf{F} = U\mathbf{v}$. We shall explore its importance at the end of this subsection. When it is added to our guess, we obtain for the total energy flux in our ideal fluid with external gravity,

$$\mathbf{F} = \rho\mathbf{v} \left(\frac{1}{2}v^2 + h + \Phi \right) , \tag{12.49}$$

where $h = u + P/\rho$ is the enthalpy per unit mass [cf. Box 12.1]. Inserting Eqs. (12.48) and (12.49) into the law of energy conservation (12.34), we get out the following ideal-fluid equation of energy balance:

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{1}{2}v^2 + u + \Phi \right) \right] + \boldsymbol{\nabla} \cdot \left[\rho\mathbf{v} \left(\frac{1}{2}v^2 + h + \Phi \right) \right] = 0 . \tag{12.50}$$

If the fluid is not ideal because heat is being injected into it by viscous heating, or being injected or removed by diffusive heat flow or by radiative cooling or by some other agent, then that rate of heat change per unit volume will be $\rho T ds/dt$, where s is the entropy per unit mass; and correspondingly, in this non-ideal case, the equation of energy balance will be changed from (12.50) to

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{1}{2}v^2 + u + \Phi \right) \right] + \boldsymbol{\nabla} \cdot \left[\rho\mathbf{v} \left(\frac{1}{2}v^2 + h + \Phi \right) \right] = \rho T \frac{ds}{dt}. \tag{12.51}$$

It is instructive and builds confidence to derive this law of energy balance from other laws, so we shall do so: We begin with the laws of mass and momentum conservation in the forms (12.25) and (12.35). We multiply Eq. (12.25) by $v^2/2$ and add it to the scalar product of Eq. (12.35) with $\rho\mathbf{v}$ to obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho v^2\mathbf{v}\right) = -(\mathbf{v}\cdot\boldsymbol{\nabla})P - \rho(\mathbf{v}\cdot\boldsymbol{\nabla})\Phi\ . \tag{12.52}$$

Assuming for simplicity that the fluid's own gravity is negligible and that the external gravitational acceleration $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$ is constant (see Box 12.2 for more general gravitational fields), we rewrite this as

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + \Phi\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + \Phi\right)\right] = -(\mathbf{v}\cdot\boldsymbol{\nabla})P\ . \tag{12.53}$$

We can now use thermodynamic identities to transform the right-hand side:

$$\begin{aligned}(\mathbf{v}\cdot\boldsymbol{\nabla})P &= \rho(\mathbf{v}\cdot\boldsymbol{\nabla})h - \rho T(\mathbf{v}\cdot\boldsymbol{\nabla})s \\ &= \boldsymbol{\nabla}\cdot(\rho\mathbf{v}h) - \rho T\frac{ds}{dt} + \rho\frac{\partial u}{\partial t} + \left(h - \frac{P}{\rho}\right)\frac{\partial\rho}{\partial t} \\ &= \boldsymbol{\nabla}\cdot(\rho\mathbf{v}h) - \rho T\frac{ds}{dt} + \frac{\partial(\rho u)}{\partial t}\ ,\end{aligned} \tag{12.54}$$

where we have used mass conservation (12.25), the first law of thermodynamics [Eq. (1) of Box 12.1] and the definition of enthalpy $h = u + P/\rho$ [Box 12.1]. Combining Eq. (12.53) with Eq. (12.54), we obtain the expected law of energy balance (12.51).

Let us return to the contribution $P\mathbf{v}$ to the energy flux. A good illustration of the necessity for this term is provided by the Joule-Kelvin method commonly used to cool gases (Fig. 12.6). In this method, gas is driven under pressure through a nozzle or porous plug into a chamber where it can expand and cool. Microscopically, what is happening is that the molecules in a gas are not completely free but attract one another through intermolecular forces. When the gas expands, work is done against these forces and the gas therefore cools. Now let us consider a steady flow of gas from a high pressure chamber to a low pressure chamber. The flow is invariably so slow (and gravity so weak!) that the kinetic and gravitational potential energy contributions can be ignored. Now as the mass flux $\rho\mathbf{v}$ is also constant the enthalpy per unit mass, h must be the same in both chambers. The actual temperature drop is given by

$$\Delta T = \int_{P_1}^{P_2}\mu_{JK}dP, \tag{12.55}$$

where $\mu_{JK} = (\partial T/\partial P)_h$ is the Joule-Kelvin coefficient. A straighforward thermodynamic calculation yields the identity

$$\mu_{JK} = -\frac{1}{\rho^2 C_p}\left(\frac{\partial(\rho T)}{\partial T}\right)_P \tag{12.56}$$

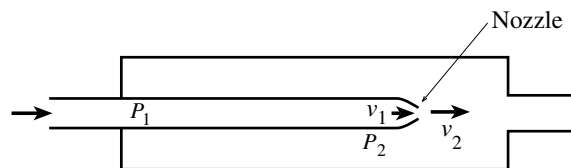The Joule-Kelvin coefficient of a perfect gas obviously vanishes.

Fig. 12.6: Joule-Kelvin cooling of a gas. Gas flows steadily through a nozzle from a chamber at high pressure to one at low pressure. The flow proceeds at constant enthalpy. Work done against the intermolecular forces leads to cooling. The efficiency of cooling is enhanced by exchanging heat between the two chambers. Gases can also be liquefied in this manner as shown here.

12.5.6 Incompressible Flows

A common assumption that is made when discussing the fluid dynamics of highly subsonic flows is that the density is constant, i.e., that the fluid is *incompressible*. This is a natural approximation to make when dealing with a liquid like water which has a very large bulk modulus. It is a bit of a surprise that it is also useful for flows of gases, which are far more compressible under static conditions.

To see its validity, suppose that we have a flow in which the characteristic length L over which the fluid variables P, ρ, v etc. vary is related to the characteristic timescale T over which they vary by $L \lesssim vT$—and in which gravity is not important. In this case, we can compare the magnitude of the various terms in the Euler equation (12.37) to obtain an estimate of the magnitude of the pressure variation $\delta P \sim \rho \delta(v^2)$. (We could just as easily have used the Bernoulli constant.) Now the variation in pressure will be related to the variation in density by $\delta P \sim c^2 \delta \rho$, where c is the *sound* speed (not light speed) and we drop constants of order unity in making these estimates. Combining these two estimates, we obtain the estimate for the relative density fluctuation

$$\frac{\delta \rho}{\rho} = \frac{\delta(v^2)}{c^2} \tag{12.57}$$

Therefore, provided that the fluid speeds are highly subsonic ($v \ll c$), then we can ignore the density variation along a streamline in solving for the velocity field. Using the equation of continuity, written as in Eq. (12.26), we can make the approximation

$$\nabla \cdot \mathbf{v} \simeq 0. \tag{12.58}$$

This argument breaks down when we are dealing with sound waves for which $L \sim cT$

It should be emphasized, though, that "incompressibility", which is an approximation made in deriving the velocity field does not imply that the density variation can be neglected in other contexts. A particularly good example of this is provided by convection flows which are driven by buoyancy as we shall discuss in Chap. 17.

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EXERCISES

Box 12.2
Self Gravity

In the text, we mostly treat the gravitational field as externally imposed and independent of the behavior of the fluid. This is usually a good approximation. However, it is inadequate for discussing the properties of planets and stars. It is easiest to discuss the necessary modifications required by self-gravitational effects by amending the conservation laws.

As long as we work within the domain of Newtonian physics, the mass conservation equation (12.25) is unaffected. However, we included the gravitational force per unit volume $\rho\boldsymbol{g}$ as a source of momentum in the momentum conservation law. It would fit much more neatly into our formalism if we could express it as the divergence of a gravitational stress tensor $\mathbf{T}_g$. To see that this is indeed possible, use Poisson's equation $\boldsymbol{\nabla} \cdot \boldsymbol{g} = 4\pi G\rho$ to write

$$\boldsymbol{\nabla} \cdot \mathbf{T}_g = -\rho\boldsymbol{g} = \frac{(\boldsymbol{\nabla} \cdot \boldsymbol{g})\boldsymbol{g}}{4\pi G} = \frac{\boldsymbol{\nabla} \cdot [\boldsymbol{g} \otimes \boldsymbol{g} - \frac{1}{2}g_e^2\mathbf{g}]}{4\pi G} \; ,$$

so

$$\mathbf{T}_g = \frac{\boldsymbol{g} \otimes \boldsymbol{g} - \frac{1}{2}g_e^2\mathbf{g}}{4\pi G} \; . \tag{1}$$

Readers familiar with classical electromagnetic theory will notice an obvious and understandable similarity to the Maxwell stress tensor whose divergence equals the Lorentz force density.

What of the gravitational momentum density? We expect that this can be related to the gravitational energy density using a Lorentz transformation. That is to say it is $O(v/c^2)$ times the gravitational energy density, where v is some characteristic speed. However, in the Newtonian approximation, the speed of light, c, is regarded as infinite and so we should expect the gravitational momentum density to be identically zero in Newtonian theory—and indeed it is. We therefore can write the full equation of motion (12.35), including gravity, as a conservation law

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T}_{\text{total}} = 0$$

where $\mathbf{T}_{\text{total}}$ includes $\mathbf{T}_g$.

Exercise 12.8 *Problem: A Hole in My Bucket*
There's a hole in my bucket. How long will it take to empty? (Try an experiment and if the time does not agree with the estimate suggest why this is so.)

Exercise 12.9 *Problem: Rotating Planets, Stars and Disks*
Consider a stationary, axisymmetric planet star or disk differentially rotating under the action of a gravitational field. In other words, the motion is purely in the azimuthal direction.

Box 12.2, Continued

Turn to energy conservation: We have seen in the text that, in a constant, external gravitational field, the fluid's total energy density U and flux $\mathbf{F}$ are given by Eqs. (12.48) and (12.49). In a general situation, we must add to these some field energy density and flux. On dimensional grounds, these must be $U_{\text{field}} \propto \boldsymbol{g}^2/G$ and $\mathbf{F}_{\text{field}} \propto \Phi_{,t}\boldsymbol{g}/G$ (where $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$). The proportionality constants can be deduced by demanding that [as in the derivation (12.52)–(12.54) of Eq. (12.51)] the laws of mass and momentum conservation imply energy conservation. The result [Ex. 12.14] is

$$U = \rho(\frac{1}{2}v^2 + u + \Phi) + \frac{g_e^2}{8\pi G} \, , \tag{2}$$

$$\mathbf{F} = \rho\mathbf{v}(\frac{1}{2}v^2 + h + \Phi) + \frac{1}{4\pi G}\frac{\partial\Phi}{\partial t}\boldsymbol{g} \, . \tag{3}$$

Actually, there is an ambiguity in how the gravitational energy is localized. This ambiguity arises physically from the fact that one can transform away the gravitational acceleration $\boldsymbol{g}$, at any point in space, by transforming to a reference frame that falls freely there. Correspondingly, it turns out, one can transform away the gravitational energy density at any desired point in space. This possibility is embodied mathematically in the possibility to add to the energy flux $\mathbf{F}$ the time derivative of $\alpha\Phi\boldsymbol{\nabla}\Phi/4\pi G$ and add to the energy density U minus the divergence of this quantity (where α is an arbitrary constant), while preserving energy conservation $\partial U/\partial t + \boldsymbol{\nabla}\cdot\mathbf{F} = 0$. Thus, the following choice of energy density and flux is just as good as Eqs. (2) and (3); both satisfy energy conservation:

$$U = \rho(\frac{1}{2}v^2+u+\Phi)+\frac{g_e^2}{8\pi G}-\alpha\boldsymbol{\nabla}\cdot\left(\frac{\Phi\boldsymbol{\nabla}\Phi}{4\pi G}\right) = \rho[\frac{1}{2}v^2+u+(1-\alpha)\Phi]+(1-2\alpha)\frac{g_e^2}{8\pi G} \, , \tag{4}$$

$$\begin{aligned}\mathbf{F} &= \rho\mathbf{v}(\frac{1}{2}v^2 + h + \Phi) + \frac{1}{4\pi G}\frac{\partial\Phi}{\partial t}\boldsymbol{g} + \alpha\frac{\partial}{\partial t}\left(\frac{\Phi\boldsymbol{\nabla}\Phi}{4\pi G}\right) \\ &= \rho\mathbf{v}(\frac{1}{2}v^2 + h + \Phi) + (1-\alpha)\frac{1}{4\pi G}\frac{\partial\Phi}{\partial t}\boldsymbol{g} + \frac{\alpha}{4\pi G}\Phi\frac{\partial\boldsymbol{g}}{\partial t} \, . \end{aligned} \tag{5}$$

[Here we have used the gravitational field equation $\nabla^2\Phi = 4\pi G\rho$ and $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$.] Note that the choice $\alpha = 1/2$ puts all of the energy density into the $\rho\Phi$ term, while the choice $\alpha = 1$ puts all of the energy density into the field term $\boldsymbol{g}^2$. In Ex. 12.11 it is shown that the total gravitational energy of an isolated system is independent of the arbitrary parameter α, as it must be on physical grounds.

A full understanding of the nature and limitations of the concept of gravitational energy requires the general theory of relativity (Part VI). The relativistic analog of the arbitrariness of Newtonian energy localization is an arbitrariness in the gravitational "stress-energy pseudotensor".

Box 12.3
Flow Visualization

There are different methods for visualizing fluid flows. We have already met the *streamlines* which are the integral curves of the velocity field **v** at a given time. They are the analog of magnetic lines of force. They will coincide with the *paths* of individual fluid elements if the flow is stationary. However, when the flow is time-dependent, the paths will not be the same as the streamlines. In general, the paths will be the solutions of the equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t). \tag{1}$$

These paths are the analog of particle trajectories in mechanics.

Yet another type of flow line is a *streak*. This is a common way of visualizing a flow experimentally. Streaks are usually produced by introducing some colored or fluorescent tracer into the flow continuously at some fixed point, say $\mathbf{x}_0$, and observing the locus of the tracer at some fixed time, say t_0. Now, if $\mathbf{x}(t; \mathbf{x}_0, t_0)$ is the expression for the location of a particle released at time t at $\mathbf{x}_0$ and observed at time t_0, then the equation for the streak emanating from $\mathbf{x}_0$ and observed at time t_0 is the parametric relation

$$\mathbf{x}(t) = \mathbf{x}(t; \mathbf{x}_0, t_0)$$

Streamlines, paths and streaks are exhibited below.
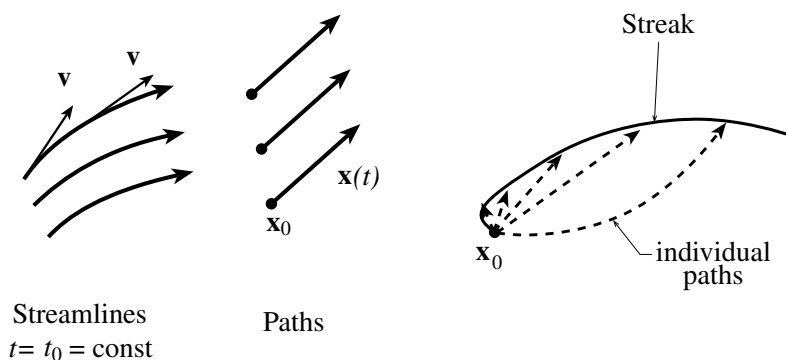
(i) Suppose that the fluid has a *barotropic* equation of state $P = P(\rho)$. Write down the equations of hydrostatic equilibrium including the centrifugal force in cylindrical polar coordinates. Hence show that the angular velocity must be constant on surfaces of constant cylindrical radius. This is called von Zeipel's theorem. (As an application, Jupiter is differentially rotating and therefore might be expected to have similar rotation periods at the same latitude in the north and the south. This is only roughly true, suggesting that the equation of state is not completely barotropic.)

(ii) Now suppose that the structure is such that the surfaces of constant entropy per unit mass and angular momentum per unit mass coincide.(This state of affairs can arise
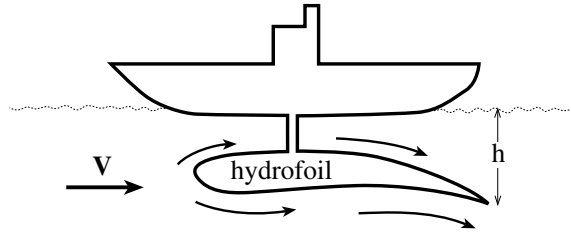
Fig. 12.7: Water flowing past a hydrofoil as seen in the hydrofoil's rest frame.

if slow convection is present.) Show that the Bernoulli function [Eq. (12.43)] is also constant on these surfaces. (Hint: Evaluate ∇B.)

Exercise 12.10 *Problem: Crocco's Theorem*
Consider steady flow of an adiabatic fluid. The Bernoulli constant is conserved along stream lines. Show that the variation of B across streamlines is given by

$$\nabla B = T\nabla s + \mathbf{v} \times \boldsymbol{\omega}$$

Exercise 12.11 *Derivation: Joule-Kelvin Coefficient*
Verify Eq. (12.56)

Exercise 12.12 *Problem: Cavitation*
A hydrofoil moves with velocity $\mathbf{V}$ at a depth $h = 3$m below the surface of a lake. (See Figure 12.7.) How fast must the hydrofoil move to make the water next to it boil? (Boiling results from the pressure P trying to go negative.)

Exercise 12.13 *Example: Collapse of a bubble*
Suppose that a spherical bubble has just been created in the water above the hydrofoil in the previous question. We will analyze its collapse, i.e. the decrease of its radius $R(t)$ from its value R_o at creation. First show that the assumption of incompressibility implies that the radial velocity of the fluid at any radial location r can be written in the form $v = F(t)/r^2$. Then use the radial component of the Euler equation (12.37) to show that

$$\frac{1}{r^2}\frac{dF}{dt} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} = 0$$

and integrate this outward from the bubble surface at radius R to infinite radius to obtain

$$\frac{-1}{R}\frac{dF}{dt} + \frac{1}{2}v^2(R) = \frac{P_0}{\rho}$$

where P_0 is the ambient pressure. Hence show that the bubble surface moves with speed

$$v(R) = \left(\frac{2P_0}{3\rho}\right)^{1/2}\left[\left(\frac{R_0}{R}\right)^3 - 1\right]^{1/2}$$

Suppose that bubbles formed near the pressure minimum on the surface of the hydrofoil are swept back onto a part of the surface where the pressure is much larger. By what factor must the bubbles collapse if they are to create stresses which inflict damage on the hydrofoil?

A modification of this solution is also important in interpreting the fascinating phenomenon of *Sonoluminescence* (Brenner, Hilgenfeldt & Lohse 2002). This arises when fluids are subjected to high frequency acoustic waves which create oscillating bubbles. The temperatures inside these bubbles can get so large that the air becomes ionized and radiates.

Exercise 12.14 *Derivation: Gravitational energy density and flux*
Show that, when the fluid with density ρ produces the gravitational field via $\nabla^2 \Phi = 4\pi G\rho$, then the law of mass conservation (12.25), the law of momentum conservation (12.35) and the first law of thermodynamics (Box 12.1) for an ideal fluid imply the law of energy conservation $\partial U/\partial t + \boldsymbol{\nabla} \cdot \mathbf{F} = 0$, where U and $\mathbf{F}$ have the forms given in Eqs. (2) and (3) of Box 12.2.

Exercise 12.15 *Example: Gravitational Energy*
Integrate the energy density U of Eq. (4) of Box 12.2 over the interior and surroundings of an isolated gravitating system to obtain the system's total energy. Show that the gravitational contribution to this total energy (i) is independent of the arbitrariness (parameter α) in the energy's localization, and (ii) can be written in the following forms:

$$
\begin{aligned}
E_g &= \int dV \frac{1}{2}\rho\Phi \\
&= -\frac{1}{8\pi G} \int dV\, g_e^2 \\
&= \frac{-G}{2} \int \int dV\, dV' \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}
\end{aligned}
$$

Interpret each of these expressions physically.

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12.6 Viscous Flows - Pipe Flow

12.6.1 Decomposition of the Velocity Gradient

It is an observational fact that many fluids develop a shear stress when they flow. Pouring honey from a spoon provides a convenient example. The stresses that are developed are known as viscous stresses. Most fluids, however, appear to flow quite freely; for example, a cup of tea appears to offer little resistance to stirring other than the inertia of the water. It might then be thought that viscous effects only account for a negligible correction to the description of the flow. However, this is not the case. Despite the fact that many fluids behave in a nearly ideal fashion almost always and almost everywhere, the effects of viscosity are still of great consequence. One of the main reasons for this is that most flows that we encounter touch solid bodies at whose surfaces the velocity must vanish. This leads to the
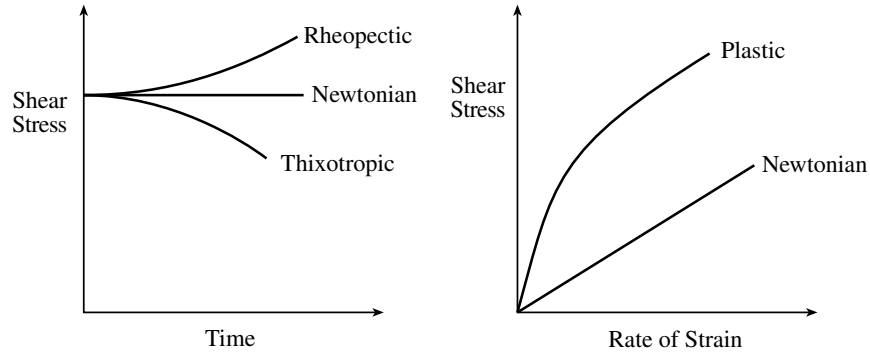
Fig. 12.8: Some examples of non-Newtonian behavior in fluids. a). In a Newtonian fluid the shear stress is proportional to the rate of shear $\boldsymbol{\sigma}$ and does not vary with time when $\boldsymbol{\sigma}$ is constant. However, some substances, such as paint, flow more freely with time and are said to be *thixotropic*. Microscopically, what happens is that the molecules become aligned with the flow which reduces the resisitance. The opposite behaviour is exhibited by *rheopectic* substances. b). An alternative type of non-Newtonian behavior is exhibited by various plastics where a threshold stress is needed before flow will commence.

formation of boundary layers whose thickness is controlled by strength of the viscous forces. This boundary layer can then exert a controlling influence on the bulk flow. It may also lead to the development of turbulence.

We must therefore augment our equations of fluid dynamics to include viscous stress. Our formal development proceeds in parallel to that used in elasticity, with the velocity field $\mathbf{v} = d\boldsymbol{\xi}/dt$ replacing the displacement field $\boldsymbol{\xi}$. We decompose the velocity gradient tensor $\boldsymbol{\nabla}\mathbf{v}$ into its irreducible tensorial parts: a *rate of expansion*, θ, a symmetric *rate of shear* tensor $\boldsymbol{\sigma}$ and an antisymmetric *rate of rotation* tensor $\mathbf{r}$, i.e.

$$\boldsymbol{\nabla}\mathbf{v} = \frac{1}{3}\theta\mathbf{g} + \boldsymbol{\sigma} + \mathbf{r}. \tag{12.59}$$

Note that we use lower case symbols to distinguish the fluid case from its elastic counterpart: $\theta = d\Theta/dt$, $\boldsymbol{\sigma} = d\Sigma/dt$, $\mathbf{r} = d\mathbf{R}/dt$. Proceeding directly in parallel to the treatment in Chap. 10 (and as already briefly sketched in Sec. 12.5.4), we write

$$\theta = \boldsymbol{\nabla} \cdot \mathbf{v}$$
$$\sigma_{ij} = \frac{1}{2}(v_{i;j} + v_{j;i}) - \frac{1}{3}\theta g_{ij}$$
$$r_{ij} = \frac{1}{2}(v_{i;j} - v_{j;i}) = -\frac{1}{2}\epsilon_{ijk}\omega^k \tag{12.60}$$

where $\boldsymbol{\omega} = d\boldsymbol{\phi}/dt$ is the vorticity, which is the counterpart of the rotation vector $\boldsymbol{\phi}$.

12.6.2 Navier-Stokes Equation

Although, as we have emphasized, a fluid at rest does not exert a shear stress, and this distinguishes it from an elastic solid, a fluid in motion can resist shear in the velocity field.

It has been found experimentally that in most fluids the magnitude of this shear stress is linearly related to the velocity gradient. This law, due to Hooke's contemporary, Isaac Newton, is the analogue of the linear relation between stress and strain that we used in our discussion of elasticity. Fluids that obey this law are known as *Newtonian*. (Some examples of the behavior of non-Newtonian fluids are exhibited in Figure 12.8.)

Fluids are usually isotropic. (Important exceptions include *smectic* liquid crystals.) Therefore, by analogy with the theory of elasticity, we can describe the linear relation between stress and rate of strain using two constants called the coefficients of *bulk* and *shear* viscosity and denoted ζ and η respectively. We write the viscous contribution to the stress tensor as

$$\mathbf{T}_{vis} = -\zeta\theta\mathbf{g} - 2\eta\boldsymbol{\sigma} \tag{12.61}$$

by analogy to Eq. (10.34).

If we add this viscous contribution to the stress tensor, then the law of momentum conservation $\partial(\rho\mathbf{v})/\partial t + \boldsymbol{\nabla}\cdot\mathbf{T} = \rho\boldsymbol{g}$ gives the following modification of Euler's equation (12.37), which contains viscous forces:

$$\rho\frac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla}P + \rho\boldsymbol{g} + \boldsymbol{\nabla}(\zeta\theta) + 2\boldsymbol{\nabla}\cdot(\eta\boldsymbol{\sigma}) \tag{12.62}$$

This is called the *Navier-Stokes equation*, and the last two terms are the viscous force density.

As we discuss shortly, it is often appropriate to ignore the bulk viscosity and treat the shear viscosity as constant. In this case, Eq. (12.62) simplifies to

$$\frac{d\mathbf{v}}{dt} = -\frac{\boldsymbol{\nabla}P}{\rho} + \boldsymbol{g} + \nu\nabla^2\mathbf{v} \tag{12.63}$$

where,

$$\nu = \frac{\eta}{\rho} \tag{12.64}$$

is known as the *kinematic viscosity*. This is the commonly quoted form of the Navier-Stokes equation.

12.6.3   Energy conservation and entropy production

The viscous stress tensor represents an additional momentum flux which can do work on the fluid at a rate $\mathbf{T}_{vis}\cdot\mathbf{v}$ per unit area. There is therefore a contribution $\mathbf{T}_{vis}\cdot\mathbf{v}$ to the energy flux, just like the term $P\mathbf{v}$ appearing in Eq. (12.51). We do not expect the viscous stress to contribute to the energy density, though.

Reworking the derivation of equation (12.51) of energy conservation, we find that we must add the term

$$v_i T_{vis,j}^{ij} = (v_i T_{vis}^{ij})_{;j} - v_{i;j}T_{vis}^{ij} \tag{12.65}$$

to Eq. (12.54). The first term of Eq. (12.65) is just the viscous contribution to the total energy flux as promised. The second term remains on the right hand side of Eq. (12.51),

which now reads

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + u + \Phi\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h + \Phi\right) - \zeta\theta\mathbf{v} - 2\eta\boldsymbol{\sigma}\cdot\mathbf{v}\right]$$
$$= \rho T\frac{ds}{dt} - \zeta\theta^2 - 2\eta\boldsymbol{\sigma}:\boldsymbol{\sigma}, \tag{12.66}$$

where $\boldsymbol{\sigma}:\boldsymbol{\sigma}$ is to be interpreted as the double contraction $\sigma_{ij}\sigma^{ij}$.

This equation needs some interpretation. Once again it is in the form of a conservation law with the rate of change of the energy density plus the divergence of the energy flux (including the viscous contribution) equaling the rate at which energy is added to the fluid. Now let us suppose that the only form of dissipation is viscous dissipation and there is no external source or sink of energy such as radiation or chemical reactions or diffusive heat flow. In this case, the total energy must be conserved without sources or sinks and the right hand side of Eq. (12.66) should vanish. Therefore, *for a viscous fluid with negligible heat flow or other sources and sinks of energy, the rate of increase of entropy due to viscous dissipation is*

$$\rho T\left(\frac{ds}{dt}\right)_{vis} = \zeta\theta^2 + 2\eta\boldsymbol{\sigma}:\boldsymbol{\sigma} , \tag{12.67}$$

the energy density is unchanged from that of an ideal fluid, $U = \rho(\frac{1}{2}v^2 + u + \Phi)$, the energy flux has the form

$$\mathbf{F} = \rho\mathbf{v}\left(\frac{1}{2}v^2 + h + \Phi\right) - \zeta\theta\mathbf{v} - 2\eta\boldsymbol{\sigma}\cdot\mathbf{v} , \tag{12.68}$$

and the law of energy conservation including viscous dissipation has the standard fundamental form $\partial U/\partial t + \boldsymbol{\nabla}\cdot\mathbf{F} = 0$.

Remarkably, we can combine the rate of viscous dissipation (12.67) with the equation of mass conservation (12.25) to obtain a conservation equation for entropy:

$$\frac{\partial(\rho s)}{\partial t} + \boldsymbol{\nabla}\cdot(\rho s\mathbf{v}) = \frac{\zeta\theta^2 + 2\eta\boldsymbol{\sigma}:\boldsymbol{\sigma}}{T} \tag{12.69}$$

The left hand side of this equation describes the rate of change of entropy density plus the divergence of entropy flux. The right hand side is therefore the rate of production of entropy. Invoking the second law of thermodynamics, this must be positive definite. Therefore the two coefficients of viscosity, like the bulk and shear moduli, must be positive.

12.6.4 Molecular Origin of Viscosity

Microscopically, we can distinguish gases from liquids. In gases, molecules of mass m travel a distance of order their *mean free path* λ before they collide. If there is a velocity gradient, $\boldsymbol{\nabla}\mathbf{v}$ in the fluid, then they will, on average, transport a momentum $\sim m\lambda\boldsymbol{\nabla}\mathbf{v}$ with themselves. If there are n molecules per unit volume traveling with mean speeds $\bar{c}$, then the extra momentum crossing a unit area in unit time is $\sim nm\bar{c}\lambda\boldsymbol{\nabla}\mathbf{v}$, from which we may extract an estimate of the coefficient of shear stress

$$\eta = \frac{1}{3}\rho\bar{c}\lambda . \tag{12.70}$$

Quantity	Kinematic viscosity ν (m^2s^{-1})
Water	10^{-6}
Air	10^{-5}
Glycerine	10^{-3}
Blood	3×10^{-6}

Table 12.2: Kinematic viscosity for common fluids.

Here the numerical coefficient of 1/3 has been inserted to agree with a proper kinetic-theory calculation. (Since, in the language of Chap. 2, the viscosity coefficients are actually "transport coefficients" for momentum, a kinetic-theory calculation can be made using the techniques of Section 2.7.) Note from Eq. (12.70) that in a gas the coefficient of viscosity will increase with temperature ($\propto T^{1/2}$).

In a liquid, where the molecules are less mobile, it is the close intermolecular attraction that dominates the shear stress. The ability of molecules to slide past one another therefore increases rapidly with their thermal activation, causing typical liquid viscosity coefficients to fall dramatically with temperature.

12.6.5 Reynolds' Number

The kinematic viscosity ν has dimensions $[L]^2[T]^{-1}$. This suggests that we quantify the importance of viscosity by comparing ν with the product of a characteristic velocity in the flow V and a characteristic length L. The dimensionless combination

$$R = \frac{LV}{\nu} \tag{12.71}$$

is known as the *Reynolds' number*, and is the first of many dimensionless numbers we shall encounter in our study of fluid mechanics. Flows with Reynolds number much less than unity – such as the tragic Boston molasses tank explosion in 1919 which caused one of the slowest floods in history – are dominated by viscosity. Large Reynolds' number flows can still be controlled by viscosity (as we shall see in later chapters), especially when acting near boundaries, despite the fact that the viscous stresses are negligible over most of the volume.

12.6.6 Blood Flow

Let us now consider one simple example of a viscous stress at work, namely the flow of blood down an artery. Let us model the artery as a cylindrical pipe of radius R, down which the blood is forced by a pressure gradient. This is an example of what is called *pipe flow*. In the absence of external forces, and time-dependence, the divergence of the total stress tensor must vanish. Therefore,

$$\mathbf{\nabla} \cdot [\rho \mathbf{v} \otimes \mathbf{v} + P\mathbf{g} - 2\eta\boldsymbol{\sigma}] = 0 \tag{12.72}$$

Now, in most instances of pipe flow $\rho v^2 \ll \Delta P =$(the pressure difference between the two ends), so we can neglect the first term in Eq. (12.72). We now suppose that the flow is solely along the $z-$ direction only a function of cylindrical radius ϖ. (This is an example

of *laminar* flow.) This is, in fact, a very important restriction. As we shall discuss in detail in the following chapter, many flows become *turbulent* and this has a major impact on the result.

As the density is effectively constant (we satisfy the conditions for incompresible flow), and we must conserve mass, the velocity cannot change along the pipe. Therefore the only non-vanishing component of the shear tensor is the ϖz component. Reexpressing Eq. (12.72) in cylindrical coordinates, and inferring from it that the pressure is a function of z only and not of ϖ, we obtain

$$\frac{1}{\varpi}\frac{d}{d\varpi}\left(\varpi\eta\frac{dv}{d\varpi}\right) = -\frac{dP}{dz} \ , \tag{12.73}$$

where dP/dz is the pressure gradient along the pipe. This differential equation must be solved subject to the boundary conditions that the velocity gradient vanish at the center of the pipe and that the velocity vanish at its walls. The solution is

$$v(\varpi) = -\frac{dP}{dz}\frac{R^2 - \varpi^2}{4\eta} \tag{12.74}$$

We can now evaluate the total discharge or mass of fluid flowing along the pipe.

$$\frac{dm}{dt} = \int_0^R \rho v 2\pi\varpi d\varpi = -\frac{\pi\rho R^4}{8\eta}\frac{dP}{dz} \tag{12.75}$$

This relation is known as *Poiseuille's law.*

Now let us apply this to blood. Consider an artery of radius $R = 1$mm. An estimate of the pressure gradient may be obtained from the difference between the diastolic and systolic pressure measured by a doctor (~ 40mm of mercury $\sim 5 \times 10^3$N m^{-2} in a healthy adult) and dividing by the length of the artery, ~ 1m. The kinematic viscosity is $\eta/\rho = \nu = 3 \times 10^{-6}$m^2 s^{-1} from Table 12.2. The rate of blood flow is then $\sim 3 \times 10^{-4}$kg s^{-1} or $\sim 3 \times 10^{-7}$m^{3}s^{-1}. Now, supposing there are ten such arteries of this size and length, the total blood flow will be $\sim 3 \times 10^{-6}$m^{3}s^{-1}.

Actually, the heart of a healthy adult pumps the full complement of blood ~ 5litres or $\sim 5 \times 10^{-3}$ m^3 every minute at a mean rate of $\sim 10^{-4}$m^{3}s^{-1} about thirty times faster than this estimate. The main reason for this large discrepancy is that we have assumed in our calculation that the walls of an artery are rigid. They are not. They are quite elastic and are able to contract and expand in a wave-like manner so as to boost the blood flow considerably. Note that the Poiseuille formula is very sensitive to the radius of the pipe, $dm/dt \propto R^4$, so a factor two increase in radius increases the flow of blood by sixteen. So, both hardening and thinning of the arteries will therefore strongly inhibit the flow of blood. Eat salads!

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EXERCISES

Exercise 12.16 *Problem: Mean free path*
Estimate the collision mean free path of the air molecules around you. Hence verify the estimate for the kinematic viscosity of air given in Table 12.2.

Exercise 12.17 *Example: Kinematic interpretation of Vorticity*
Consider a velocity field with non-vanishing curl. Define a locally orthonormal basis at a
point in the velocity field so that one basis vector, $\mathbf{e}_x$ is parallel to the vorticity. Now imagine
the remaining two basis vectors as being frozen into the fluid. Show that they will both rotate
about the axis defined by $\mathbf{e}_x$ and that the vorticity will be the sum of their angular velocities
(i.e. twice the average of their angular velocities).

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